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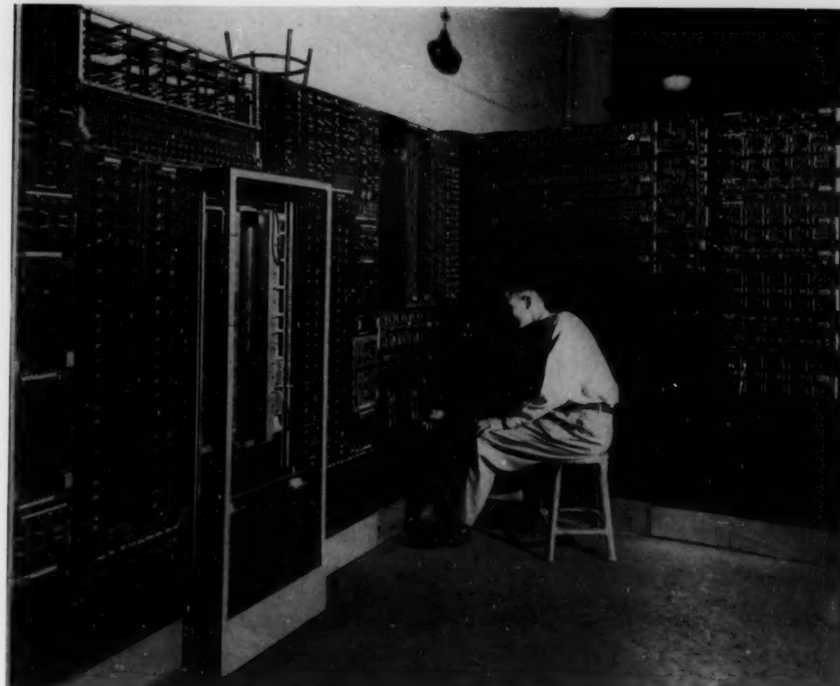

CALIFORNIA DIGITAL COMPUTER

Sept. 1950

The California Digital Computer

The California Digital Computer (CALDIC) is to be operated by the Engineering Department of the University of California at Berkeley for the benefit of its own research and that of the other departments and for the instruction of graduate students. This fact has strongly affected its logical design (in the hope of avoiding a large highly-trained staff) and also its physical design (requiring all parts to be accessible for demonstration and servicing). It is strictly a low-cost machine and makes no claims to superiority in speed or originality in design; most of the electronic techniques used are well established. It is not, however, designed for maximum simplicity of construction but rather for simplicity of description to and operation by the varied group of scientists who are expected to bring their computing to it. Its operation is to be straight-forward and as nearly like the familiar hand computational methods as possible, at least until experience is gained in its use. Its construction permits easy modification in the future, and it is expected that such modification will occur.

The frontispiece shows most of the portions of the computer which are completed at this writing (September 1, 1950). At the left is the memory unit, consisting of a vertical rotating drum with four columns of combination reading-recording heads arranged around it and a set of electronic panels approximately six feet high by ten feet wide which contain the circuitry required to write into and read from the drum. To the right of these is the first of the arithmetic registers; the second has been built but not yet installed on the blank panels below. At the extreme right is the order counter which is a shifting and counting register designed to store and supply the address of the next order to be carried out. The panels shown, a few of which are temporary, were built first to allow their operation as a test system, and, as other parts of the computer are completed, they will be added to the system.

Figure 1 shows a simplified block diagram of the computer. It is a serial machine so far as decimal numbers are concerned; but the binary parts of each digit are transmitted in parallel, and most of the circuits have four channels. The three arithmetic registers are of the shifting type, with circulation paths provided as shown. The "A" or accumulator register incorporates an adder in its first column, and circulation of its contents simultaneously with receipt of a new number causes the sum to appear and stand in the register. For multiplication the "D" register holds the multiplicand, which is repeatedly circulated but simultaneously shifted and added into the A register as required by the digits of the multiplier which are drawn from the "R" register. As the multiplier digits are used the overflow from the A register appears in their places, and at the end of the process the double-length product occupies both the A and the R registers.

All timing pulses for the computer are drawn from a timing track permanently recorded on the drum, and therefore no effort to control the drum speed is needed. A small gap in the timing track is used to provide an "origin pulse," and recorded words are located by counting the timing pulses, beginning the count at the origin. Information recorded on the drum is therefore not lost during power failures or other shut-downs; subroutines could

be permanently stored if desired. The timing pulses are divided by suitable circuits into "digit pulses" in groups of eleven which provide for shifting numbers through the registers and "space pulses" which occur between the groups of digit pulses and which initiate all switching.

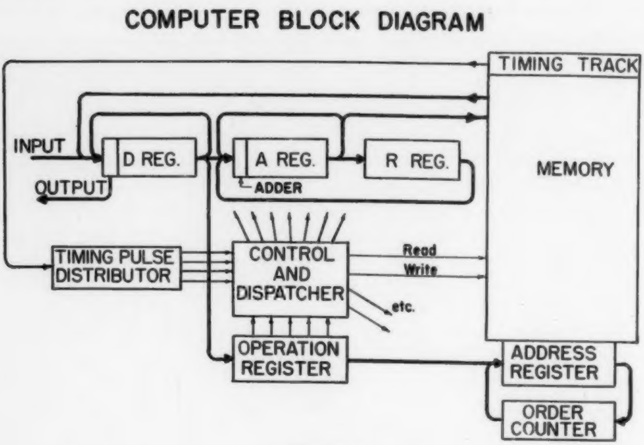

FIGURE 1

The operation of the computer will be cyclic, essentially as follows:

1. The address of the next order to be performed is shifted from the order counter to the address register and used to extract the order from the memory into the D register.

2. The order is shifted into the operation and address registers, and simultaneously the contents of the address register are returned to the order counter and increased by one to provide for the next cycle.

3. The address register now contains the address of the operand, which is withdrawn from the memory into the D register and operated on as required by the contents of the operation register. At the end of the cycle the result stands in the A register.

The normal cycle requires two trips to the memory, each of which might take almost the time of a full drum revolution (17 milliseconds) in addition to the time required for the operation itself which is usually small in comparison. In most cases, however, the orders will be in consecutive memory boxes and the whole cycle will require only slightly more than the time of one revolution. In addition, the consecutively-numbered boxes are spaced around the drum at intervals which correspond to the time required for an addition and order-shift, and recognition of this fact in programming may make it possible to carry out as many as ten operations per revolution of the drum. Abnormal cycles are used for several orders for which no new operand is needed; shift-left, shift-right, and sub-program orders are of this type. The cycle time for these, however, is about the same as for normal cycles. Multiplication and division, which are performed by repeated addition, may

require an additional revolution. Because of these variations the average operation speed is difficult to estimate, but it should be considerably better than 50 single-address operations per second on most problems.

The orders to be provided are: add, subtract, multiply, multiply and round off, divide, divide and round off, extract the square root, read the input tape, transfer to memory, print out, clear address and add, change contents of order counter, change contents of order counter but only if the previous operation overflowed, shift left, and shift right.

The only unusual orders are the "clear and add" and the conditional sub-program orders. The former provides for the modification of orders by building the desired address in the A register and attaching the operation part of the order to it. The latter provides for discrimination and also for overflow of the accumulator; a possible overflow is provided for by inserting

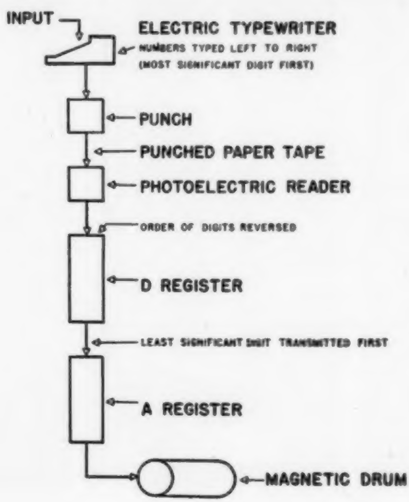

FIGURE 2

the instruction which tells the computer what to do if the overflow occurs. If no overflow occurs, the instruction is disregarded.

Orders consist of two decimal digits which specify the operation and four additional digits which usually specify the location in the memory of the operand, but sometimes they give other information such as the location of a sub-routine or the number of places of shifting in a shift-left order. No use is made of the sign column or the remaining four digit positions of the normal 10-decimal-digit word.

Input will be from perforated tape, using the system diagrammed in Figure 2. In general all orders and data will be stored in the memory at the beginning of a problem, and the input will not be consulted further during the computation. The "read the input" order is designed mainly to instruct the machine to proceed to the next problem when one is completed; the tape reader will be loaded manually, and the tape will be run clear through on

receipt of the "read" order. The information read from the tape may, however, be placed in any desired portions of the memory, as specified by addresses punched into the tape ahead of the blocks of information. The entire memory can be filled in approximately three minutes. Output also will be through punched tape followed by printing with a standard tape-reading typewriter. Output speed is limited to punch speed of about ten digits per second.

The outstanding feature of the computer is its magnetic memory, shown in more detail in Figure 3. The drum is about eight inches in diameter and

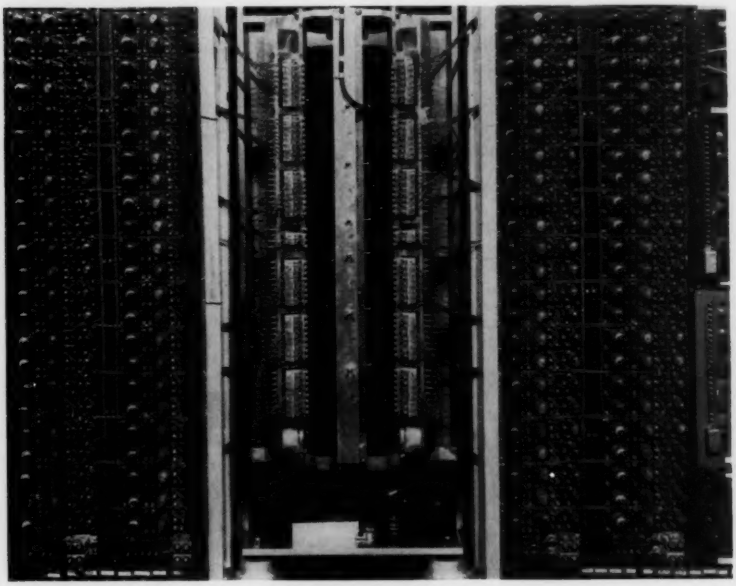

FIGURE 3. Magnetic Memory

26 inches long and will store 10,000 ten-digit numbers with their signs. Since the no-excess binary code is used, over 480,000 memory cells are required at a density of about 900 per square inch of surface. There are 200 specially-designed magnetic reading-recording heads, and each is provided with a writing tube, a writing gate, and a reading gate. In addition 50 tubes are used in the band switch, and with the timing circuits and the 4-channel reading and writing amplifier circuits the memory unit requires about 750 tube envelopes, many of them double triodes. This is more than half of the approximately 1300 tubes used in the whole computer along with 1000 crystal diodes.

Practically all of the design and construction of the CALDIC is the work of graduate and undergraduate students in Electrical Engineering, some 35 having been associated with the project already. The electronic work is now being supplemented by research and instruction on programming and applications. The need for both types of training is demonstrated by the demand

which has developed for graduates of the program. Financial support from the Mathematics Branch of the Office of Naval Research has made the work possible, and techniques developed by other ONR-supported projects have been used freely.

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Step-by-Step Integration of $\ddot{x} = f(x, y, z, t)$ without a "Corrector"

Introduction.—For step-by-step numerical integration of ordinary differential equations there are too many formulae, too few evaluations or comparisons. Perhaps the complexity of the subject will not permit the generalizations the mathematician would prefer, but only restricted numerical comparisons of specific procedures. This paper, at any rate, will restrict itself to showing reasons for preferring one of two procedures for the integration of second-order differential equations in which the second derivative, $\ddot{x} = \dfrac{d^2x}{dt^2}$, is a function of x alone, or of x and t, or of x, y, z, t (with similar equations for $\ddot{y}$ and $\ddot{z}$). These equations are important, of course, in certain dynamical problems of astronomy, ballistics, aerodynamics, etc., including the rocket problem.

The comparison of the two procedures will extend to at least three equivalent forms, in which the integration formula at each step is based upon

(a) antecedent values of $\ddot{x}$,
(b) backwards differences of $\ddot{x}$,
(c) central differences (estimated) of $\ddot{x}$.

Although there are reasons for preferring forms (a) or (b) in certain circumstances, the comparison will be made first between the procedures in form (c) because there is some advantage to distinguishing between errors of estimation and errors resulting from the neglect of higher order terms in the integration formulae.

The preferred procedure, designated the "second-sum procedure" or "Σ^2 procedure," involves tables of $\ddot{x}$ and its first and second sums, $\Sigma\ddot{x}$ and $\Sigma^2\ddot{x}$, as well as its differences (explicitly or implicitly), $\delta\ddot{x}$, $\delta^2\ddot{x}$, $\delta^3\ddot{x}$, $\cdots$. It has been used for more than a century by astronomers,[1,2] but has apparently been overlooked by a number of mathematicians, physicists, and others, in recent years.

The compared procedure, designated the "second-difference procedure" or "δ^2 procedure," involves the summation (explicit or implicit) of the second difference of x, δ^2x, which is obtained by formula from $\ddot{x}$ and its differences (or the equivalent). This procedure was used by COWELL & CROMMELIN, perhaps for the first time in an extended calculation, in their celebrated prediction of the return of Halley's Comet in 1910. In publishing the results of this work[3] they recommended without explanation, however, that future integrations of this type should be done with the Σ^2 procedure and formulae. In spite of this recommendation a great deal of work has been done by the

δ^2 procedure, by astronomers as well as others, that could have been shown to be better adapted to the Σ^2 procedure.

In short this paper will show that (1) under comparable circumstances, the δ^2 procedure requires two approximations (or the use of "predictor" and "corrector" formulae[8]) whereas the Σ^2 procedure requires only one. It will show further that (2) the estimated differences normally employed by the astronomer may be replaced by backwards differences or antecedent values of the second difference without increasing the error or requiring a second approximation. The second of these facts, as well as the first, is significant to modern calculation with automatic electronic digital computers.

The δ^2 Procedure.—For the purposes of the comparison it will be necessary first to outline the "δ^2 procedure" with estimated central differences, since these may be unfamiliar to many persons otherwise well acquainted with the problem. Their introduction makes it possible to use the same formula for "predictor" and "corrector," the difference between the two approximations being due to the improvement in the estimates of the differences involved. This formula[7] we shall call the "(δ_c^2) formula" and write as follows:

$$(\delta_c^2) \qquad \delta^2 x_n = h^2 \left\{ \ddot{x}_n + \frac{1}{12}\,\delta^2\ddot{x}_n - \frac{1}{240}\,\delta^4\ddot{x}_n + \frac{31}{60480}\,\delta^6\ddot{x}_n - \cdots \right\}$$

where h is the interval of the argument (0.1 in the table below), the notation for the differences is best appreciated by reference to the numerical table below, and the subscript n indicates that the various quantities are to be found on the same line of the table (the underlined quantities on the 0.8 line in the example below).

The following table shows an integration table for the simple differential equation $\ddot{x} = -x$. How the table was started is not significant; at the given step all numbers not in parentheses may be taken as correct. The values of the differences of $\ddot{x}$ in parentheses are estimates based upon the assumption that the sixth difference is zero. The estimates of $\delta^2 x_{0.8}$, $\delta x_{0.85}$, and $x_{0.9}$, on the other hand, are a result of the first approximation or "prediction" below the table.

t	x	δx	$\delta^2 x$	$\ddot{x} = -x$	$\delta\ddot{x}$	$\delta^2\ddot{x}$	$\delta^3\ddot{x}$	$\delta^4\ddot{x}$	$\delta^5\ddot{x}$
0.0	+.000 000 000		000 000	−.000 0000		0000		000	
		+99 833 417			−99 8334		+9975		−100
0.1	+.099 833 417		−997 502	−.099 8334		+9975		−100	
		+98 835 915			−98 8359		+9875		−97
0.2	+.198 669 332		−1985 038	−.198 6693		+19850		−197	
		+96 850 877			−96 8509		+9678		−100
0.3	+.295 520 209		−2952 740	−.295 5202		+29528		−297	
		+93 898 137			−93 8981		+9381		−91
0.4	+.389 418 346		−3890 939	−.389 4183		+38909		−388	
		+90 007 198			−90 0072		+8993		−89
0.5	+.479 425 544		−4790 261	−.479 4255		+47902		−477	
		+85 216 937			−85 2170		+8516		−89
0.6	+.564 642 481		−5641 721	−.564 6425		+56418		−566	
		+79 575 216			−79 5752		+7950		(−89)
0.7	+.644 217 697		−6436 810	−.644 2177		+64368		(−655)	
		+73 138 406			−73 1384		(+7295)		(−89)
0.8	+.717 356 103		($\underline{-7167\ 586}$)	$\underline{-.717\ 3561}$		($\underline{+71663}$)		($\underline{-744}$)	
		(+65 970 820)							
0.9	(+.783 326 923)								

		"Prediction"	"Correction"
$+$	$\ddot{x}_{0.8} =$	$-0.717\ 3561$	$-0.717\ 3561$
$+$	$\frac{1}{12}\delta^2\ddot{x}_{0.8} =$	$+5971\ .9$	$+5973\ .0$
$-$	$\frac{1}{240}\delta^4\ddot{x}_{0.8} =$	$+3\ .1$	$+3\ .0$
	$h^{-2}\delta^2 x_{0.8} =$	$-0.716\ 7586\ .0$	$-0.716\ 7585\ .0$
	$\delta^2 x_{0.8} =$	$-.007167586$	$-.007167585$

After the predicted $\delta^2 x_{0.8}$ is summed in the table to produce an estimate of $x_{0.9}$, it is possible to calculate accurately $\ddot{x}_{0.9} = -0.783\,3269$. [From an inspection of the "correction" we may verify that no significant revision of this number will be necessary.] The table may now be differenced to show that the accurate value of $\delta^2\ddot{x}_{0.8} = +71676$, and this number replaces the estimate ($+71663$) in the "correction," with a resulting change of one digit in $\delta^2 x_{0.8}$, $\delta x_{0.85}$, and $x_{0.9}$. The table is now ready for the next step.

The Σ^2 Procedure.—The "(Σ_c^2) formula" is as follows:

$$(\Sigma_c^2) \qquad x_n = h^2\left\{\Sigma^2\ddot{x}_n + \frac{1}{12}\ddot{x}_n - \frac{1}{240}\delta^2\ddot{x}_n + \frac{31}{60480}\delta^4\ddot{x}_n - \cdots\right\}$$

where the notation is the same as in the (δ_c^2) formula. The numerical coefficients are the same in the two formulae because the one is simply the second sum or the second difference of the other. The table associated in this procedure with the numerical example of the preceding section is as follows:

t	$\Sigma^2\ddot{x}$	$\Sigma\ddot{x}$	$\ddot{x} = -x$	$\delta\ddot{x}$	$\delta^2\ddot{x}$	$\delta^3\ddot{x}$	$\delta^4\ddot{x}$	$\delta^5\ddot{x}$
0.0	+0.000 0000		−.000 0000		0000		000	
		+9.991 6653		−99 8334		+9975		−100
0.1	+9.991 6653		−.099 8334		+9975		−100	
		+9.891 8319		−98 8359		+9875		−97
0.2	+19.883 4972		−.198 6693		+19850		−197	
		+9.693 1626		−96 8509		+9678		−100
0.3	+29.576 6598		−.295 5202		+29528		−297	
		+9.397 6424		−93 8981		+9381		−91
0.4	+38.974 3022		−.389 4183		+38909		−388	
		+9.008 2241		−90 0072		+8993		−89
0.5	+47.982 5263		−.479 4255		+47902		−477	
		+8.528 7986		−85 2170		+8516		−89
0.6	+56.511 3249		−.564 6425		+56418		−566	
		+7.964 1561		−79 5752		+7950		(0)
0.7	+64.475 4810		−.644 2177		+64368		(−566)	
		+7.319 9384		−73 1384		(+7384)		(0)
0.8	+71.795 4194		−.717 3561		(+71752)		(−566)	
		+6.602 5823		(−65 9632)		(+6818)		
0.9	+78.398 0017		(−.783 3193)		(+78570)			

NOTE 1: The estimates of the differences (in parentheses) are arbitrarily made poorer in this example than in that of the preceding section, to emphasize the fact that an even larger error of estimation will not require a second approximation in the Σ^2 procedure.

NOTE 2: (For those unfamiliar with sum columns.) Given a starting value, the numbers in the $\Sigma\ddot{x}$ column are summed from the $\ddot{x}$ column just as the $\ddot{x}$ column could be summed from the $\delta\ddot{x}$ column. Similarly the $\Sigma^2\ddot{x}$ column is summed from the $\Sigma\ddot{x}$ column. Starting values may be obtained by an inversion of the process followed below.

NOTE 3: The subscript n refers to the 0.9 line in this example rather than the 0.8 line, as indicated by the underlining, in order to make the figures of the two examples more directly comparable.

The calculation of $x_{0.9}$ and of an accurate value of $\ddot{x}_{0.9}$ to replace the estimate of this quantity proceeds as follows, in accordance with equation (Σ_c^2):

$$
\begin{array}{rrl}
+ & \Sigma^2\ddot{x}_{0.9} = & +78.398\quad 00\ 2 \\
+\frac{1}{12} & \ddot{x}_{0.9} = & \quad\ -65\quad 27\ 7 \\
-\frac{1}{240} & \delta^2\ddot{x}_{0.9} = & \qquad\qquad -3\ 3 \\
\hline
& h^{-2}x_{0.9} = & +78.332\quad 69\ 2 \\
\ddot{x}_{0.9} = & -x_{0.9} = & -\ 0.783\,3269
\end{array}
$$

Although this value of $\ddot{x}_{0.9}$ differs by 76 units of the last place from the estimate in the table, it will be found, in checking the figures of a possible "correction" or recalculation, that the preliminary value of $\frac{1}{12}\ddot{x}_{0.9}$, and hence the final value of $\ddot{x}_{0.9}$, will not be altered from what is shown in the above calculation. Nor would they if an even greater error were made in the estimate.

Comparison.—1. It will be observed, in the numerical work of the two examples, that the $\frac{1}{12}$ term involves in the first a 5 digit number ($\delta^2\ddot{x}_{0.8} = +71663$), in the second a 7 digit number ($\ddot{x}_{0.9} = -7833193$), but that only 5 digits are carried in either calculation ($+59719$ and -65277). That is in the Σ^2 procedure the last two digits of the estimate of $\ddot{x}$ are of no significance, and a much larger estimation error can be tolerated than in the $\delta^2\ddot{x}$ that plays the same role in the δ^2 procedure. Thus the first approximation is sufficient in the Σ^2 procedure, but a second is necessary in the δ^2 procedure to alter or check the last digit in the calculation. It is assumed, it will be noted, that successive columns of sums, function, and differences taper off in size by about one digit or more per column. When the tapering is more, the situation is even more favorable to the Σ^2 procedure: for example, if there are four more digits in the $\Sigma^2\ddot{x}$ column than in the $\ddot{x}$ column, the last four digits of the estimate of $\ddot{x}$ will not be significant, and a very crude estimate is tolerable. A tapering of one digit per column is probably the least that would be tolerated in any procedure for numerical integration when automatic or rote methods are employed; an experienced computer, using the utmost of judgment, may be able to integrate by these procedures when the tapering is only one digit per two columns.

2. Since a larger estimation error can be tolerated in the Σ^2 procedure than in the δ^2 procedure, it is evident that fewer difference columns will need to be taken into account, an advantage when the memory of an automatic computer is limited.

3. It is assumed that the accumulation of rounding error and the effect of neglected terms will be about the same in the two procedures. The discussion is concerned instead with the effect of errors of estimation, explicitly as in the foregoing example or implicitly in equivalent processes involving backwards differences or antecedent values of $\ddot{x}$. (The estimation of differences should not be confused with a first approximation or "prediction,"

but should be recognized as merely an alternative way of making the integration depend upon backwards differences.)

4. It is assumed that the purpose of numerical integration is the construction of as accurate as possible a table from which the integral may be obtained, and not necessarily a table of the integral itself. Often, as in comet orbits or ballistics tables, only a few values of the integral at the end of the table are needed. Since, however, it is customary to carry two or three more places in a table than are required in the end, it will be noted that the values of the integral obtained in the first approximation by either procedure are probably sufficiently accurate for all intended uses. From this point of view, evidently the second approximation in the δ^2 procedure is necessary only to avoid the accumulation of error in the summation of $\delta^2 x$. The equivalent summation of $\ddot{x}$ in the Σ^2 procedure is free of estimation error. For interpolating non-tabular values of the integral, BOWER has supplied a table.[8]

5. To shorten the paper, I have not included in the foregoing discussion a third procedure which may be used in integration of second-order differential equations. This procedure is based upon the formula for the second derivative, $\ddot{x}$, in terms of the differences of x, which we may write

$$\delta^2 x = h^2 \ddot{x} + \frac{1}{12}\delta^4 x - \frac{1}{90}\delta^6 x + \cdots.$$

My investigation of this procedure revealed only a greater disadvantage: namely, three approximations as compared with the two of the δ^2 procedure and the one of the Σ^2 procedure. Various attempts have been made to improve this formula for restricted classes of second-order differential equations by introducing special devices to eliminate the $\frac{1}{12}$ term—beginning with NUMEROV,[4] and most recently by FOX & GOODWIN.[9] At best these devices yield results comparable to the Σ^2 procedure only when the 1/240 term is negligible, and at least for some of them JACKSON[5] is right in believing that they would become the Σ^2 procedure if they were carried to their logical conclusion.

Another proposal sometimes made in connection with the second-derivative formula above is that it be used to correct a first approximation in which the estimation error is ignored. This proposal would have merit if it were not for the fact that the estimation error is negligible in the first approximation by the Σ^2 procedure.

The Backwards-Difference Formula.—When the central differences used in the (Σ_c^2) formula are estimated by a summation based upon the repetition of the last value of a given difference, the calculation will yield exactly the same result as if it had been based upon a backwards-difference formula ending with the same difference. This formula is as follows:

$$(\Sigma_b^2) \qquad x_n = h^2 \left\{ \Sigma^2 \ddot{x}_n + \frac{1}{12}\left[\ddot{x}_{n-1} + \delta \ddot{x}_{n-3/2} + \frac{19}{20}\delta^2 \ddot{x}_{n-2} + \frac{18}{20}\delta^3 \ddot{x}_{n-5/2} + \frac{1726}{2016}\delta^4 \ddot{x}_{n-3} + \frac{1650}{2016}\delta^5 \ddot{x}_{n-7/2} + \cdots \right] \right\}.$$

The estimation of differences may thus be entirely avoided in rote calculation, by automatic machinery or otherwise.

The Antecedent-Function Formula.—When the computer feels that he can dispense with the check afforded by the run of the differences, he may replace the (Σ_b^2) formula by the equivalent expression in terms of antecedent values of the function to be integrated:

$$(\Sigma_a^2) \qquad x_n = h^2 \left\{ \Sigma^2\ddot{x}_n + \frac{1}{12}\left(1 + 1 + \frac{19}{20} + \frac{18}{20} + \cdots\right)\ddot{x}_{n-1} \right.$$
$$- \frac{1}{12}\left(1 + 2\cdot\frac{19}{20} + 3\cdot\frac{18}{20} + \cdots\right)\ddot{x}_{n-2}$$
$$+ \frac{1}{12}\left(\frac{19}{20} + 3\cdot\frac{18}{20} + \cdots\right)\ddot{x}_{n-3}$$
$$- \frac{1}{12}\left(\frac{18}{20} + \cdots\right)\ddot{x}_{n-4}$$
$$\left. + \cdots \right\}.$$

It is evident that this general expression will yield a number of formulae, depending upon the order of the last difference of the (Σ_b^2) formula that is taken into account:

$$(\Sigma_a^2)_0 \qquad x_n = h^2\left\{\Sigma^2\ddot{x}_n + \frac{1}{12}\ddot{x}_{n-1}\right\}$$

$$(\Sigma_a^2)_1 \qquad x_n = h^2\left\{\Sigma^2\ddot{x}_n + \frac{1}{12}(2\ddot{x}_{n-1} - \ddot{x}_{n-2})\right\}$$

$$(\Sigma_a^2)_2 \qquad x_n = h^2\left\{\Sigma^2\ddot{x}_n + \frac{1}{240}(59\ddot{x}_{n-1} - 58\ddot{x}_{n-2} + 19\ddot{x}_{n-3})\right\}$$

$$(\Sigma_a^2)_3 \qquad x_n = h^2\left\{\Sigma^2\ddot{x}_n + \frac{1}{240}(77\ddot{x}_{n-1} - 112\ddot{x}_{n-2} + 73\ddot{x}_{n-3} - 18\ddot{x}_{n-4})\right\}$$

and so on, where $(\Sigma_a^2)_m$ includes the effect of $\delta^m\ddot{x}$.

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[1] T. R. VON OPPOLZER, *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, v. 2, Leipzig, 1880.

[2] J. C. WATSON, *Theoretical Astronomy*. Philadelphia, 1892.

[3] P. H. COWELL, & A. C. D. CROMMELIN, *Investigation of the Motion of Halley's Comet from 1759 to 1910*. Appendix to *Greenwich Observations* for 1909. Edinburgh, 1910. p. 84.

[4] B. V. NUMEROV, *Méthode nouvelle de la détermination des orbites et le calcul des éphémérides en tenant compte des perturbations*. Observatoire Astrophysique Central de Russie, *Publications*, v. 2, Moscow, 1923. "A method of extrapolation of perturbations," Roy. Astr. Soc., *Monthly Notices*, v. 84, 1924, p. 592–601.

[5] J. JACKSON, Note on the numerical integration of $\frac{d^2x}{dt^2} = f(x, t)$. Roy. Astr. Soc., *Monthly Notices*, v. 84, 1924, p. 602–606.

[6] E. C. BOWER, "Coefficients for interpolating a function directly from a table of double integration." Lick Observatory, *Bull.*, no. 445, v. 16, 1932, p. 42.
[7] *Interpolation and Allied Tables*, reprinted from the British Nautical Almanac for 1937, 4th edition with additions. (London, H. M. Stationery Office, 1946.)
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Formulas for Calculating the Error Function of a Complex Variable

1. Various methods have been suggested for the computation, to high accuracy, of the error function $\Phi(Z) = \int_0^Z e^{-u^2}du$ for complex arguments $Z = X + iY$. The Maclaurin series is convenient for extreme accuracy only when $|Z|$ is small, and the asymptotic expansions[1] are useful only for fairly large values of $|Z|$. Other less elementary methods (e.g., the AIREY converging factor,[2] or the continued fraction expansion[1]) have limited success in certain regions of the Z-plane. The most convenient methods for calculating $\Phi(Z)$ to very many figures are described in the works of MILLER & GORDON[3] and ROSSER.[4] It is the purpose of this note, which is self-contained, to present in a concise and practical form, two schemes for the computation of $\Phi(Z)$ which follow the main ideas of Miller, Gordon, and Rosser.

2. The basis of the present methods is the following formula:

$$(1)\qquad \sum_{n=-\infty}^{\infty} \exp\left(-(u+na)^2\right) = \pi^{\frac{1}{2}}a^{-1} \sum_{n=-\infty}^{\infty} \exp\left(-n^2\pi^2a^{-2}\right) \cos\left(2n\pi u/a\right)$$

which is an immediate corollary of POISSON's formula.[5] Formula (1) is also essentially Jacobi's imaginary transformation,[6] which has long been familiar to a number of mathematicians and physicists.[7,8] Also, (1) was used by DAWSON[9] in his computation of $\int_0^Y e^{u^2}du$. Two different methods for calculating $\Phi(Z)$ are given here in **4** and **5**. In all formulas, summations are from 1 to ∞ except where otherwise noted. The symbol $\doteq$ will denote approximate equality.

3. Dawson's formula for $\int_0^Y e^{u^2}du$ is needed in the second method. It follows from the approximate equality obtained from (1), which is

$$(2)\qquad e^{u^2}(1+E) = a\pi^{-\frac{1}{2}}\left(1 + 2\sum \exp\left(-a^2n^2\right)\cosh 2nau\right), \qquad a \leqslant 1$$

where the relative error $E = 2\sum e^{-n^2\pi^2/a^2}\cos(2n\pi u/a)$, is of the order of magnitude of $2e^{-\pi^2/a^2}$, since $2e^{-4\pi^2/a^2}$ is very small by comparison. Approximate values of E are given in the following table:

a	1	0.9	0.8	0.7	0.6	0.5
E	10^{-4}	10^{-5}	$\frac{3}{2}\cdot10^{-6}$	$\frac{1}{2}\cdot10^{-8}$	$\frac{1}{4}\cdot10^{-11}$	$\frac{3}{2}\cdot10^{-17}$

Putting $a = \frac{1}{2}$ in (2) Dawson obtains

$$(3)\qquad e^{u^2} \doteq \pi^{-\frac{1}{2}}\left(\tfrac{1}{2} + \sum e^{-n^2/4}\cosh nu\right),$$

the relative error in (3) being less than $2\cdot10^{-17}$. Then integration of (3) results in

$$(4)\qquad \int_0^Y e^{u^2}du \doteq \pi^{-\frac{1}{2}}\left(\tfrac{1}{2}Y + \sum n^{-1}\exp\left(-n^2/4\right)\sinh nY\right).$$

4. One method of obtaining a formula for $\Phi(Z)$ is to choose the path of integration from 0 to Z to consist of the line segments $[0, X]$ and $[X, X + iY]$. We find

$$(5)\quad \Phi(Z) = \int_0^X e^{-u^2}du + e^{-X^2}\int_0^Y e^{u^2}\sin 2Xu\,du + ie^{-X^2}\int_0^Y e^{u^2}\cos 2Xu\,du.$$

The first term on the right is numerically well known.[10] To deal with the other terms, we multiply across in (3) by $\sin 2Xu$ and $\cos 2Xu$ respectively, and integrate between 0 and Y. Using the explicit expressions for integrals of the form $\int \cosh nu \begin{Bmatrix}\sin qu\\ \cos qu\end{Bmatrix} du$ we readily obtain $\Phi(Z) = A + iB$, where

$$(6)\quad A \doteq \int_0^X e^{-u^2}du + \pi^{-\frac{1}{2}}e^{-X^2}(1 - \cos 2XY)/4X + \pi^{-\frac{1}{2}}e^{-X^2}\sum (n^2 + 4X^2)^{-1}(2X - 2X\cosh nY\cos 2XY + n\sinh nY\sin 2XY)\cdot\exp(-n^2/4)$$

and

$$(7)\quad B \doteq \pi^{-\frac{1}{2}}e^{-X^2}((4X)^{-1}\sin 2XY + \sum (n^2 + 4X^2)^{-1}\exp(-n^2/4)\cdot\{2X\cosh nY\sin 2XY + n\sinh nY\cos 2XY\}).$$

5. Another formula for calculating $\Phi(Z)$ is found by choosing the path of integration to consist of the line segments $[0, iY]$ and $[iY, X + iY]$, when we find

$$(8)\quad \Phi(Z) = i\int_0^Y e^{u^2}du + e^{Y^2}\int_0^X e^{-u^2}\cos 2Yu\,du - ie^{Y^2}\int_0^X e^{-u^2}\sin 2Yu\,du.$$

The first term on the right can be found from Dawson's approximation in (4). The other two terms are found by letting $a = 12$ in (1), then multiplying across by $\cos 2Yu$ and $\sin 2Yu$, and integrating between 0 and X. In view of the fact that $\int_6^\infty e^{-u^2}du$ is negligible, being of order 10^{-17}, it is easily shown that if we neglect the integral of everything arising from $n \neq 0$ in the left member of (1), when $X \leqslant 6$ the error is $< 2\int_6^\infty e^{-u^2}du$. When $X > 6$, we can replace X by 6 in (8), with a relative error $< \int_6^\infty e^{-u^2}du$, and in view of the result in the preceding sentence, the total error is surely $< 3\int_6^\infty e^{-u^2}du$. For maximum accuracy it is not necessary to take more than 24 terms arising from under the summation sign in the right member of (1), because then the truncating error, which is of the order of $e^{-4\pi^2}$, is less than the error in the approximation formula. Making use of the explicit expressions for integrals of the form $\int \cos pu \begin{Bmatrix}\sin qu\\ \cos qu\end{Bmatrix} du$, whenever $|Y| \neq n\pi/12$, we obtain $\Phi(Z) = C + iD$, where

$$(9)\quad C \doteq \frac{\pi^{\frac{1}{2}}}{12}e^{Y^2}((2Y)^{-1}\sin 2XY + 72\sum_{n=1}^{24}(n^2\pi^2 - 144Y^2)^{-1}\exp(-n^2\pi^2/144)\cdot\{\tfrac{1}{6}\pi n\sin\tfrac{1}{6}\pi nX\cos 2XY - 2Y\cos\tfrac{1}{6}\pi nX\sin 2XY\})$$

and

$$(10)\quad D \doteq \pi^{-\frac{1}{2}}\Big(\tfrac{1}{2}Y + \sum n^{-1}\exp(-n^2/4)\sinh nY\Big) - \frac{1}{12}\pi^{\frac{1}{2}}e^{Y^2}\Big((2Y)^{-1}(1-\cos 2XY) + 72\sum_{n=1}^{24}(n^2\pi^2 - 144Y^2)^{-1} \times \exp(-n^2\pi^2/144)\Big\{\frac{n\pi}{6}\sin \tfrac{1}{6}n\pi X \sin 2XY + 2Y\cos \tfrac{1}{6}\pi n X \cos 2XY - 2Y\Big\}\Big).$$

In the exceptional case when $|Y| = n\pi/12$ for some value of n, omit that term from inside the summation signs in C and D and instead add

$$(11)\quad \pi^{\frac{1}{2}}(48Y)^{-1}(4XY + \sin 4XY - i\{1 - \cos 4XY\}).$$

Both methods which are described in **4** and **5** are accurate to within a relative error of about 10^{-16} in $|\Phi(Z)|$, except for values of $|Y|$ very close to $n\pi/12$ in the latter method. The amount of computational work seems to be about the same for these two methods, and in general it appears difficult to recommend one of the pairs of formulas (6), (7) or (9), (10) in preference to the other. Each method affords a good computational check upon the use of the other.

6. Often the value of $e^{Z^2}\Phi(Z)$ is desired because it is a smoother function which can be tabulated to a constant number of decimals. Thus from (6) and (7) we have at once $e^{Z^2}\Phi(Z) = E + iF$, where

$$(12)\quad E \doteq \exp(X^2 - Y^2)\cos 2XY\int_0^X e^{-u^2}du + \pi^{-\frac{1}{2}}e^{-Y^2}\Big((4X)^{-1}(\cos 2XY - 1) + 2X\sum(n^2+4X^2)^{-1}\exp(-n^2/4)\{\cos 2XY - \cosh nY\}\Big)$$

and

$$(13)\quad F \doteq \exp(X^2 - Y^2)\sin 2XY\int_0^X e^{-u^2}du + \pi^{-\frac{1}{2}}e^{-Y^2}\{(4X)^{-1}\sin 2XY + \sum(n^2+4X^2)^{-1}\exp(-n^2/4)(2X\sin 2XY + n\sinh nY)\}.$$

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[4] J. B. ROSSER, *Theory and Application of* $\int_0^x e^{-x^2}dx$ *and* $\int_0^x e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$. Part I. *Methods of Computation*, New York, 1948.

[5] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*. Oxford, 1937, p. 60–64.

[6] E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*. Fourth ed., Cambridge, 1940, p. 124, 474–476.

[7] E. T. GOODWIN, "The evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 241–245.

[8] A. M. Turing, "A method for the calculation of the Zeta-function," London Math. Soc., *Proc.*, s. 2, v. 48, 1943, p. 180–197.
[9] H. G. Dawson, "On the numerical value of $\int_0^h e^{x^2}dx$," London Math. Soc., *Proc.*, s. 1, v. 29, 1898, p. 519–522.
[10] NBS, *Tables of Probability Functions*. V. 1, New York, 1941.

RECENT MATHEMATICAL TABLES

857[**D**].—J. K. Lynch, "Calculation of $(\sin x)/x$ tables," Postmaster-General's Department, *Research Laboratory Report No. 3238*, Melbourne, C. 1., (April 1949), C.E. 615/R, Case no. 2266. 7 mimeographed pages + drawing.

These tables of $(\sin x)/x$ were computed to facilitate the calculation of the harmonic components of periodic impulses. The function $(\sin x)/x$ is tabulated for $x = 0(1°)180°(5°)1800°$; 4D, with first differences. The drawing contains a graph of $(\sin x)/x$ for x ranging from 0° to 1600°, together with a straight line converting degrees to radians. The statement in the Introduction ". . . the interval of tabulations is such that linear interpolation will suffice," is not quite correct, since second differences are needed for full accuracy when x lies between 180° and 295°.

For another table of this function, see [*MTAC*, v. 4, p. 80].

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858[**F, G, O**].—R. E. Beard, "On the coefficients in the expansion of e^{e^t} and e^{-e^t}." Inst. of Actuaries, *Jn.*, v. 76, 1950, p. 152–163.

The expansions mentioned in the title are written

$$e^{e^t} = e \sum_{n=0}^{\infty} K_n t^n/n!$$

$$e^{-e^t} = e^{-1} \sum_{n=0}^{\infty} A_n t^n/n!$$

where the A's and K's are integers. They are tabulated for $n = 0(1)26$. [See the following review where K_n is denoted by $U(n)$ and a table is described for $n = 1(1)50$; the agreement for $n \leqslant 26$ is exact.] The author used the formulas

$$K_r = \sum_{k=1}^{r} \Delta^k 0^r/k!$$

and

$$A_r = \sum_{k=1}^{r} (-1)^k \Delta^k 0^r/k!$$

The values of the Stirling numbers of the second kind, $\Delta^k 0^r/k!$ were taken from Fisher & Yates [as corrected in *MTAC*, v. 4, p. 27] for $r \leqslant 25$. The necessary additional values of these numbers for $r = 26$ are appended on p. 163. The table of A_r appears to be new. Previous tables of K_n for $n \leqslant 20$ are mentioned in FMR, *Index* § 4.676.

D. H. L.

859[F, G, O].—H. GUPTA, "Tables of distributions," East Panjab University, *Research Bull.*, no. 2, 1950, p. 13–44.

The author denotes by $u(n, a)$ the number of ways that n unlike objects may be distributed into a boxes. If $a > n$ then $u(n, a) = 0$. The sum

$$U(n) = \sum_{a=1}^{n} u(n, a)$$

is the total number of distributions of n different objects into groups. It also represents the number of rhyming schemes in a stanza of n lines. In terms of generating functions we have

$$\sum_{n=0}^{\infty} u(n, a)x^n/n! = (e^x - 1)^a/a!, \qquad \sum_{n=0}^{\infty} U(n)x^n/n! = e^{-1} \exp(e^x).$$

The fundamental difference equation for $u(n, a)$ is

$$u(n + 1, a) = au(n, a) + u(n, a - 1).$$

By means of this relation the author has constructed a table (p. 17–43) of $u(n, a)$ for $1 \leqslant a \leqslant n \leqslant 50$, and has obtained, by summing, a table of $U(n)$ for $n = 1(1)50$. The function $U(n)$ may also be computed from the relation

$$U(n + 1) = \sum_{k=0}^{n} U(k)\binom{n}{k}. \tag{1}$$

This was used to check the whole set of tables by computing the 48-digit number $U(50)$. The several terms of (1) for $n = 50$ are set forth on p. 44.

D. H. L.

860[F].—H. GUPTA, "A table of values of Liouville's function $L(t)$," East Panjab University, *Research Bulletin*, no. 3, 1950, p. 45–63.

The function of LIOUVILLE, despite the title of the paper under review, is usually denoted by $\lambda(n)$ and is $+1$ or -1 according as the number of prime factors of n, each reckoned with its multiplicity, is even or odd. The function $L(t)$ is then defined as the sum function

$$L(t) = \sum_{n \leqslant t} \lambda(n).$$

According to a conjecture of POLYA, the values of $L(t)$ for $t > 1$ are never positive. This conjecture, when proved, will imply the truth of the famous RIEMANN hypothesis. In 1940 the author constructed a table of $L(t)$ for $t = 1(1)20000$ [*MTAC*, v. 1, p. 201]. The present paper presents a condensation of this table from which with very little effort an isolated value in the original table may be found. More specifically, the present table gives $-L(t)$ for $t = 5(5)20000$ and, by a clever device, the values of $\lambda(t)$ when t is not a multiple of 5. ($\lambda(5k) = -\lambda(k)$.)

Two small tables (p. 47) give some idea of the range of values of $L(t)$ for $t \leqslant 20000$. The conjecture of Polya is verified but nevertheless $-L(t)$, though positive, is fairly small. A maximum of 150 occurs at $t = 15810$. The function $t^{-1}L^2(t)$ has a maximum at $t = 9840$. This would indicate that $L(t) = O(t^{\frac{1}{2}})$ which would also imply the Riemann hypothesis. The author

mentions that isolated values of $L(t)$ up to $t = 60000$ may be found by another method. By still another method the reviewer has found that $L(4000000)$ is very close to -1100.

D. H. L.

861[F].—D. JARDEN, "On the numbers V_{5n} (n odd) in the sequence associated with Fibonacci's sequence," *Riveon Lemat.*, v. 4, 1950, p. 38–40 [Hebrew with English summary].

This note contains a factor table of the numbers

$$V_{2n} - 5U_n + 3 \qquad \text{and} \qquad V_{2n} + 5U_n + 3$$

whose product, in case n is odd, is V_{5n}/V_n. Here V_m is the mth term of the FIBONACCI sequence: 2, 1, 3, 4, 7, $\cdots$

$$V_m = V_{m-1} + V_{m-2}, \qquad 5U_m = V_m + 2V_{m-1}.$$

The table is for $n = 1(2)77$. Factorization is incomplete in many cases. For the function $V_{2n} - 5U_n + 3$ complete factorization into primes is given only for $n = 1(2)35, 39(2)45, 49, 51, 63, 75$; for the second function only for $n = 1(2)45$. Most of the results come from the table of KRAITCHIK.[1]

D. H. L.

[1] M. KRAITCHIK, *Recherches sur la Théorie des Nombres.* Paris, 1924, v. 1, p. 80.

862[K].—A. N. BLACK, "Weighted probits and their use," *Biometrika*, v. 37, 1950, p. 158–167.

The paper tabulates

$$P = (2\pi)^{-\frac{1}{2}} \int_{\infty}^{Y-5} \exp\left(-\tfrac{1}{2}u^2\right)du, \qquad w = Z^2(PQ)^{-1},$$

$$wy_0 = w(Y - PZ^{-1}), \qquad wy_m = w(Y + QZ^{-1})$$

all to 3D for $Y = 1(.02)9$, except that wy_0 is replaced by $100 - wy_0$ for $Y > 6.42$. As usual $P + Q = 1$ and $Z = (2\pi)^{-\frac{1}{2}} \exp\left(-(Y-5)^2/2\right)$. The purpose of the table is to minimize the arithmetic of the maximum likelihood estimation of the probit regression. There are detailed instructions for calculation.

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863[K].—F. N. DAVID, "Note on the application of Fisher's k-statistics," *Biometrika*, v. 36, 1949, p. 383–393.

This paper contains a table of the terms which must be added to $\kappa(r^h s^l)$, the product cumulant of order hl of the r-th and s-th k-statistics about the population r-th and s-th cumulants respectively in terms of population cumulants, in order to give the corresponding product moment. The table gives all terms for $h + l = 4$ and 5 and those for $h + l = 6$ involving n^{-3} only where n is the sample size. The paper illustrates the utility of these tables in reducing the algebra in finding the approximate moment characteristics of certain statistics.

C. C. C.

864[K].—F. N. DAVID, "Two combinatorial tests of whether a sample has come from a given population," *Biometrika*, v. 37, 1950, p. 97–110.

In connection with the tests herein described and illustrated, the author tabulates (Table 1a, p. 109) $N^{-N}\binom{N}{Z}\Delta^{N-Z}0^N$ for $N = 3(1)20$ and $Z = 0(1)19$ to 4D. This is the probability of Z zero groups when N sample units are randomly arranged in N groups of equal probability and is thus related to a classical distribution problem. Table 4a, p. 110 gives values of $(2N)^{-N}\binom{N}{Z}\Delta^{N-Z}(N^N)$, for $N = 1(1)10$ and $Z = 0(1)10$ to 4D, which is the probability that if N balls be dropped at random into $2N$ compartments, Z of a specified set of N compartments will be empty.

C. C. C.

865[K].—F. E. GRUBBS, "Sample criteria for testing outlying observations," *Annals Math. Stat.*, v. 21, 1950, p. 27–58.

The author proposes the following statistics

$$\frac{S_n^2}{S^2} = \frac{\sum_{i=1}^{n-1}(x_i - \bar{x}_n)^2}{\sum_{i=1}^{n}(x_i - x)^2} = 1 - \frac{1}{n-1}T_n^2, \qquad T_n = \frac{x_n - \bar{x}}{S}$$

$$\frac{S_1^2}{S^2} = \frac{\sum_{i=2}^{n}(x_i - \bar{x}_1)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = 1 - \frac{1}{n-1}T_1^2, \qquad T_1 = \frac{\bar{x} - x_1}{S}$$

where $x_1 \leq x_2 \leq \cdots \leq x_n$, $\bar{x}_n = \frac{1}{n-1}\sum_{i=1}^{n-1} x_i$, $\bar{x}_1 = \frac{1}{n-1}\sum_{i=2}^{n} x_i$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

for testing the greatest and lowest observations respectively in a sample of size n, from a normal population. Values of S_n^2/S^2 or S_1^2/S^2 to 4D are given at percentage points 1, 2.5, 5 and 10 for $n = 3(1)25$ in Table I (p. 28) and corresponding values to 3D of T_n or T_1 in Table IA (p. 29). Grubbs' values in Table IA calculated from the exact distribution concur with those of E. S. PEARSON & C. CHANDRA SEKAR[1] for common entries.

Table II (p. 31–37) contains values of $P(u_n \leq u)$ to 5D for $u = 0(.05)4.85$, $n = 2(1)17$, $u = .5(.05)4.90$, $n = 18$, 19 and to 4D for $u = .5(.05)4.90$, $n = 20(1)25$, where $u_n = (x_n - \bar{x})/\sigma$ in samples of size n drawn from a parent normal population with zero mean and standard deviation σ. These values were obtained from the cumulative distribution

$$F_n(u) = n^{\frac{1}{2}}(2\pi\{n-1\})^{-\frac{1}{2}}\int_0^u \exp\left(-\frac{x^2}{2}n/(n-1)\right)F_{n-1}(nx/(n-1))dx$$

where $F_n(u) = \int_0^u dF(u_n) = P(u_n \leq u)$. This distribution (obtained in 1945) was also independently arrived at by K. R. NAIR[2] and is equivalent to the result of A. T. MCKAY[3] arrived at by means of characteristic functions. Nair[2] also gave tables of this probability integral for $u = 0(.01)4.7$,

$n = 2(1)9$, to 6D which concur in corresponding values with Grubbs' Table II.

Table III (p. 45) contains values to 3D of $u_n = (x_n - \bar{x})/\sigma$ for percentage points 90, 95, 99, and 99.5, $n = 2(1)25$, also given by Nair[2] to 2D for percentage points .01, .5, 1, 2.5, 5, 10, 90, 95, 97.5, 99, 99.5, 99.9, $n = 3(1)9$.

Table IV (p. 46) contains the mean, standard deviation, α_3 and α_4 of u_n to 4D for $n = 2(1)15$, and to 3D for $n = 20, 60, 100, 200, 500$, and 1000. These values agree with corresponding values of TIPPETT[4] for the mean and those of McKay[3] for the standard deviation except for $n = 2$, which Grubbs gives correctly as 0.4263 (by McKay's formula) whereas McKay gives 0.6179.

Table V contains percentage points 1, 2.5, 5, and 10, $n = 4(1)20$ to 4D for $S^2_{n-1,n}/S^2$ or $S^2_{1,2}/S^2$ where

$$S^2_{n-1,n} = \sum_{i=1}^{n-2} (x_i - \bar{x}_{n-1,n})^2, \qquad \bar{x}_{n-1,n} = (n-2)^{-1} \sum_{i=1}^{n-2} x_i,$$

$$S^2_{1,2} = \sum_{i=3}^{n} (x_i - \bar{x}_{1,2})^2, \qquad \bar{x}_{1,2} = (n-2)^{-1} \sum_{i=3}^{n} x_i,$$

S^2 and $\bar{x}$ as above. These statistics are proposed for testing the significance of the two largest and the two smallest observations in samples of size n drawn from parent normal populations.

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[1] E. S. PEARSON & C. CHANDRA SEKAR, "The efficiency of statistical tools and a criterion for the rejection of outlying observations," *Biometrika*, v. 28, 1936, p. 308–320.

[2] K. R. NAIR, "The distribution of the extreme deviate from the sample mean and its studentized form," *Biometrika*, v. 35, 1948, p. 118–144.

[3] A. T. MCKAY, "The distribution of the difference between the extreme observation and the sample mean in samples of n from a normal universe," *Biometrika*, v. 27, 1935, p. 466–471.

[4] L. H. C. TIPPETT, "On the extreme individuals and the range of samples taken from a normal population," *Biometrika*, v. 17, 1925, p. 364–387.

866[K].—A. HALD, "Maximum likelihood estimation of the parameters of a normal distribution which is truncated at a known point," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 119–134.

This paper includes four short tables (p. 130–134) of certain functions for use in estimating the mean and variance of normal populations from one-sided truncated samples with known points of truncation, both for the simple truncated case in which the number of missing (unmeasured) observations is unknown and for the case which the author labels as "censored" in which the number of unmeasured observations is known. Tables I and II pertain to the simple truncated case. Tables III and IV furnish similar information for the "censored" case. All four tables were computed from Fisher's I or Hh function.[1]

Table I of $z = f(y)$ is an inversion of Table XVI from the B.A.A.S. Tables[1] where z is the abscissa of truncation in standard units of the complete distribution and y is a function of moments of the truncated distribution. In practice, one sets $y = n \sum x_i^2/2(\sum x_i)^2$ and reads z directly from

this table. Entries of z are given to 3D for $y = .55(.001).91$. In the notation of PEARSON & LEE,[2] $y = (\psi, -1)/2$, and in the notation of FISHER[3] $y = I_0 I_2/I_1^2$. Table II contains $g(z)$, a function employed in estimating the variance, together with its first differences as an aid in interpolation. Entries are tabulated to 4D(4 or 5S). This table also includes elements of the variance-covariance matrix, $\mu_{11}(z)$, $\mu_{12}(z)$, and $\mu_{22}(z)$ tabulated to 4S, and $\rho(z)$, the correlation coefficient between estimates of population parameters, to 3D. The argument, z, ranges over $-3(.1)2$.

Table III of $z = f(h, y)$ was also computed by Statistika I/S for $h = .55(.05).8$. It permits the abscissa of truncation, z, for the "censored" case to be read directly for given values of h and y, where h is the ratio of the number of unmeasured to measured observations, and y is the counterpart of the similarly labeled function for the simple truncated case. Entries for z are given to 3D with $y = .5(.001)1.5$ and $h = .05(.05).8$. The functions of Table IV for the "censored" case correspond to those of Table III for the simple truncated case. The range and the interval of the argument, z, are the same in both tables, and the number of significant digits is the same in most instances.

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[1] B.A.A.S., *Math. Tables*, v. 1, 2nd ed., Cambridge, 1946, Tables XV and XVI.
[2] KARL PEARSON & ALICE LEE, "On the generalized probable error in multiple normal correlation," *Biometrika*, v. 6, 1908, p. 59–68.
[3] See 1st ed. of above B.A.A.S. tables, 1931, p. xxvi–xxxv.

867[K].—H. O. HARTLEY & E. S. PEARSON, "Table of the probability integral of the *t*-distribution," *Biometrika*, v. 37, 1950, p. 168–172.

This table gives the cumulative probability integral of *Student's t* for γ degrees of freedom, namely

$$P(t, \gamma) = \left\{\Gamma\left(\frac{\gamma+1}{2}\right)\Big/[\Gamma(\tfrac{1}{2}\gamma)\sqrt{\pi\gamma}]\right\}\int_{-\infty}^{t}(1 + x^2/\gamma)^{-(\gamma+1)/2}dx$$

to 5D for $\gamma = 1(1)20$; $t = 0(.1)4(.2)8$ and $\gamma = 20(1)24, 30, 40, 60, 120, \infty$; $t = 0(.05)2(.1)4, 5$. Also values of t and α such that $1 - P(t, \gamma) = \alpha$ are given for $\alpha = 10^{-3}, 10^{-4}, 10^{-5}, 5(10)^{-6}$; $\gamma = 1(1)10$. For $\gamma = 11(1)20$ the values of α to 10^{-5}, as well as values of α to $3(10)^{-6}$ and less for larger γ, are covered in the main part of the table. Formulas for both single-entry and double-entry interpolation are given.

This table may be particularly useful in problems where it is necessary to interpolate for fractional degrees of freedom. It is also useful in cases where distributions of other statistics can be expressed in terms of the t-distribution. Previous tables[1] were to 4D for $\gamma = 1(1)20$.

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[1] *Student*, "New tables for testing the significance of observations," *Metron*, v. 5, no. 3, 1925, p. 105–120.

868[L].—C. J. Bouwkamp, (I) "On the characteristic values of spheroidal wave functions," *Philips Res. Rep.*, v. 5, 1950, p. 87–90. (II) "On the theory of spheroidal wave functions of order zero," Nederl. Akad. Wetensch., *Proc.*, v. 53, 1950, p. 931–944.

I. The differential equation for spheroidal wave functions is taken in the form

$$(1) \qquad (1 - z^2)y'' - 2zy' + (\lambda - m^2(1 - z^2)^{-1} + k^2z^2)y = 0.$$

Corresponding to a countably infinite set of characteristic values λ, for given m and k, equation (1) admits a solution which is regular at $z = \pm 1$. It is known that the form for λ is

$$\lambda_n^m(k) = \sum_{i=0}^{\infty} p_i k^{2i}.$$

The author gives the coefficients p_i, $i = 0(1)4$, for general n and m, along with numerical values of the coefficients (in both fractional and decimal form) for $m = 1$, $n = 1(1)7$.

Three brief tables are given:

Table 1: Prime factors in the denominators of the coefficients p_i, for $n = 1(1)7$; $m = 1$.

Table 2: Decimal form of the coefficients p_i for $n = 1(1)7$; $m = 1$, to 12D. Meixner (1944) gave a more complete table, to 10D. [*MTAC*, v. 3, p. 524–6.]

Table 3: Numerical values of $\lambda_n^1(k)$, as calculated from power series, for $k^2 = \pm 1, \pm 4$, $n = 1(1)7$; to 8D for $k^2 = \pm 1$, and 5D for $k^2 = \pm 4$.

II. In this paper the author treats various series representations for the solutions of the first, second, and third kinds, associated with the characteristic values of the equation. In particular he derives a set of coefficients c_n, such that two independent solutions are of the form

$$y(z) = \exp(-kz) \sum_{n=0}^{\infty} c_n P_n(z); \qquad Y^*(z) = \exp(-kz) \sum_{n=0}^{\infty} c_n Q_n(z),$$

where $P_n(z)$ and $Q_n(z)$ are the Legendre functions. Here $y(z)$ is the regular solution. These solutions are valid in the domain of regularity of the function (outside the branch cut joining $z = 1$ and $z = -1$). Interesting relations are obtained between the eigenvalues $\mu_n(k)$ of the integral equation

$$y(z) = \mu \int_{-1}^{1} \exp(kzt) y(t) dt$$

and the coefficients c_p. Although the series involving c_p are important theoretically they do not converge as rapidly as the classical series involving Bessel functions.

Gertrude Blanch

NBSINA

869[L].—B. W. Conolly, "A short table of the confluent hypergeometric function $M(\alpha, \gamma, x)$," *Quart. Jn. Mech. Appl. Math.*, v. 2, 1950, p. 236–240.

Tables of confluent hypergeometric functions are listed in section 22.56 of the FMR *Index*, and some have also been reviewed in *MTAC*. In the

present paper the author tabulates

$$M(\alpha, \gamma, x) = \sum_{r=0}^{\infty} \Gamma(\gamma)\Gamma(\alpha + r)x^r/(r!\Gamma(\alpha)\Gamma(\gamma + r)) = \sum_{r=0}^{\infty} a_r x^r$$

to 11D for $x = .1, .2(.2)1$, $\alpha = -1(.2)1$, and $\gamma = .2(.2)1$. The tabulated values are believed to be correct to within 2 or 3 units of the last decimal place.

The coefficients a_r, and the values of $M(\alpha, \gamma, 1)$ were provided by J. C. P. MILLER.

A. E.

870[L].—A. COOMBS, "The translation of two bodies under the free surface of a heavy fluid." Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 453–468.

In the course of the work a 6D table is constructed of

$$\{\mathrm{li}(e^{\alpha}) - \pi i\}\exp(-\alpha(1 - \tfrac{1}{2}i\beta)) - \sum_{n=0}^{\infty} \alpha^{-n-1}n! \sum_{s=n+1}^{\infty} (\tfrac{1}{2}i\alpha\beta)^s/s!$$

for $\beta = -1, 1$ and $\alpha = 1, 2, 2.5, 3(1)6, 8$. More extensive tables are provided for the forces acting upon two cylinders in the same vertical or horizontal plane.

A. E.

871[L].—OTTO EMERSLEBEN, "Die elektrostatische Gitterenergie eines neutralen ebenen, insbesondere alternierenden quadratischen Gitters." *Zeit. für Physik*, v. 127, 1950, p. 588–609.

In the study of crystal lattices there appear the EPSTEIN zeta functions, defined for $R(s) > 2$ by

$$Z\begin{vmatrix} 0 & 0 \\ h_1 & h_2 \end{vmatrix}(s) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty}{}' \frac{\exp(2\pi i(k_1h_1 + k_2h_2))}{(k_1^2 + k_2^2)^{s/2}}.$$

For various integer values of s the author evaluates these functions in the four cases $(h_1, h_2) = (0, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4})$. The last three cases can be computed directly from the first with the help of functional equations. For $s = 1$ the formula

$$Z\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}(1) = -4 + 2\sum_{n=1}^{\infty} n^{-\frac{1}{2}}r_2(n)[1 - F((\pi n)^{\frac{1}{2}})]$$

is used, where $r_2(n)$ is the number of ways n can be written as a sum of two squares and

$$F(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-t^2}dt.$$

Eight terms of the series suffice to yield the value

$$Z\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}(1) = -3.900\,264\,92.$$

For $s = 2$ the values have been previously given by the author.[1] For $s = 3$

and $s = 4$ use is made of the product representation

$$Z\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}(s) = 4\zeta(\tfrac{1}{2}s)L(\tfrac{1}{2}s)$$

and tables of $\zeta(s)$, $L(s)$. Here

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad L(s) = \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)^{-s}.$$

An alternative method is also given in the case $s = 4$. For negative integer values of s the author uses the functional equation which gives the value at s in terms of the value at $2 - s$ and gamma functions.

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[1] OTTO EMERSLEBEN, "Einige Identitäten für Epsteinsche Zetafunktionen," *Math. Ann.*, v. 121, 1949, p. 103–106.

872[L].—B. FRIEDMAN, "Theory of underwater explosion bubbles," *Comm. on Pure and Applied Math.*, v. 3, 1950, p. 177–199.

Tables are given to 4S of the functions

$$f(x) = 2x \sum_{n=0}^{\infty} (-1)^n[(2n+1)^2 - x^2]^{-1}$$

$$F(x) = (1 - x)[\tfrac{1}{2}x^{-1}f(x) + \tfrac{1}{2}\ln 2 - (1 + x)^{-1}].$$

The range for x is 0(.05)1 for f and -1(.05)1 for F.

873[L].—E. L. KAPLAN, "Multiple elliptic integrals," *Jn. Math. Phys.*, v. 29, 1950, p. 69–75.

The author shows that the evaluation of integrals of the form

$$\int K\, k^n k'^{n'} dk, \qquad \int E\, k^n k'^{n'} dk$$

where n, n' are integers and k, k', K, E are the standard notations, and the evaluation of some similar integrals can be reduced to the computation of the eight integrals

$$\int K\, dk/k, \quad \int K'\, dk/k, \quad \int K\, dk/k', \quad \int E\, dk/k',$$

$$\int K\, dk, \quad \int K'\, dk, \quad \int K\, dk/k'', \quad \int K'\, dk/k'',$$

and that the last four can be dispensed with if Landen's transformation is used in the process of reduction. The values of the 8 basic integrals taken between the limits 0 and 1, and modified if necessary to produce convergence, can be expressed in terms of known constants and are given in the paper to 10D. There is, moreover, a 10D table of

$$(n + 1) \int_0^1 k^n K\, dk$$

for $n = -.9(.1)2$.

A. E.

874[L].—A. Leitner & R. D. Spence, "The oblate spheroidal wave functions." Franklin Institute, *Jn.*, v. 249, No. 4, 1950, p. 299–321 [see also ref. in *MTAC*, v. 3, 1948, p. 99–101].

The differential equations for the "angular" and "radial" spheroidal wave functions of order m respectively are written in the form

$$(1) \qquad d\{(1-\eta^2)du/d\eta\}/d\eta - \{m^2(1-\eta^2)^{-1} - \alpha + \epsilon^2(1-\eta^2)\}u = 0,$$

$$(2) \qquad d\{(1+\xi^2)dv/d\xi\}/d\xi - \{m^2(1+\xi^2)^{-1} - \alpha + \epsilon^2(1+\xi^2)\}v = 0.$$

Here $\epsilon = ka$ (k is a given parameter and a is the radius of the focal circle) and α is a separation constant whose eigenvalues α_{lm} are such that a solution u_{lm} belonging to α_{lm} is finite at $\eta = \pm 1$.

After a summary of the methods for representing the eigenvalues α_{lm} and the functions $u_{lm}(\eta)$ and $v_{lm}(\xi)$, expansion formulas for spherical and plane waves in terms of the spherical wave functions are given. The numerical results consist of tables and curves [p. 314–320] and contain for $\epsilon = 1(1)5$ the following quantities:

The eigenvalues α_{lm} for $m = 1$, $l = 4(1)8$; $m = 2$, $l = 2(1)9$; $m = 3$, $l = 4(1)10$; $m = 4$, $l = 4(1)11$.
The norm N_{lm} for $m = 0$, $l = 0(1)5$; $m = 1$, $l = 1(1)6$.
The norm N_{lm} is defined as $\int_{-1}^{1} u_{lm}u_{l'm}d\eta = \delta_{ll'}N_{lm}$.
q_{lm} for $m = 0$, $l = 0(1)5$; $m = 1$, $l = 1(1)6$
Q_{lm} for $m = 0$, $l = 0(1)5$.
The q_{lm} and Q_{lm} are constants associated with the radial functions.
Finally *the angular function* $u_{lm}(\eta)$ for $m = 0$, $l = 0(1)5$; $m = 1$, $l = 1(1)6$; $m = 2$, $l = 2$ and 3, and for values of the argument $\eta = 0(0.1)1$.

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875[L].—Brigitte Radon, "Sviluppi in serie degli integrali ellittici," Accad. Naz. Lincei, *Atti, Mem., Cl. Sci. Fis. Mat. Nat.*, s. 8, v. 2, 1950, p. 69–109.

The usual notations of elliptic integrals,

$$F(\varphi, k) = \int_0^\varphi \Delta^{-1}(\psi, k)d\psi, \qquad E(\varphi, k) = \int_0^\varphi \Delta(\psi, k)d\psi,$$

$$\pi(\varphi, \rho, k) = \int_0^\varphi [(1 + \rho \sin^2 \psi)\Delta(\psi, k)]^{-1}d\psi$$

are used, where

$$0 \le \varphi \le \tfrac{1}{2}\pi, \qquad 0 \le k^2 < 1, \qquad \text{and} \qquad \Delta(\psi, k) = (1 - k^2 \sin^2 \psi)^{\frac{1}{2}}.$$

The author gives the expansion of both complete ($\varphi = \frac{1}{2}\pi$) and incomplete (φ general) integrals in powers of k^2, also in powers of $k'^2 = 1 - k^2$ (in the latter case there are also logarithmic terms). In addition, for the incomplete integrals there are expansions in powers of $\Delta(\varphi, k) - \cos \varphi$, in powers of $1 - \Delta(\varphi, k)$, and for F and E also as a trigonometric series. For the integrals of the third kind expansions in powers of $k^2 + \rho$ are also provided.

In each case the first few coefficients are given explicitly, for the others either a general form or a recurrence relation. The region of convergence is noted, as is the region of "practical convergence" where the series converges at least as well as an infinite geometrical progression with ratio $\frac{1}{2}$.

A. E.

876[L].—Wasao Shibagaki, *Table of the Modified Bessel Functions*, Kyushu University, Physics Department, April 1946, 132 p. 25.4 × 18.3 cm.

This table tabulates the modified Bessel functions $I_n(x)$, $K_n(x)$ or $\sqrt{x}\, I_n(x)$, $\sqrt{x}\, K_n(x)$.

The functions $I_0(x), I_1(x), \cdots, I_{21}(x)$ are tabulated for $x = 0(.01)1(.02)5$; $I_{22}(x)$ for $x = 1(.02)5$; the functions $x^{\frac{1}{2}}I_n(x)$ are tabulated for $x = 5(.04)25$, $n = 0(1)22$.

The functions $K_n(x)$ are tabulated for $x = 0(.01)1(.02)5$, $n = 0(1)23$; the functions $x^{\frac{1}{2}}K_n(x)$ are tabulated for $x = 5(.04)25$, $n = 0(1)23$.

All functions are given to mostly five significant figures, and to facilitate interpolation the first differences, divided by 100 h, where h denotes the tabular interval in x, are tabulated alongside the functions. The purpose is to avoid the computation of $p = \Delta x/h$ in the interpolation.

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877[L].—Tokyo Numerical Computation Bureau, *Report no.* 3. *Tables of Spherical Wave Functions*, 1950. VI. 1. (Keisuh-Renkyukjo) 466, Kyohdoh, Setagaya, Tokyo, Japan. 1950, iv, 77 p. 25.4 × 36.2 cm.

The text (p. i–iv) of this publication is in Japanese although the title page is in English. Various tables occupy p. 2–77. The tables are primarily concerned with the Riccati Bessel functions arising in connection with the wave equation, as follows:

$$
\begin{aligned}
F_n(x) &= (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x) = x^{n+1} f_n(x)/(2n+1)!! \\
G_n(x) &= (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{-n-\frac{1}{2}}(x) = x^{-n} g_n(x)(2n-1)!! \\
F_n'(x) &= (n+1)x^n \bar{f}_n(x)/(2n+1)!! \\
G_n'(x) &= n x^{-n-1} \bar{g}_n(x)(2n-1)!!
\end{aligned}
$$

where $(2n+1)!! = 1\cdot 3\cdot 5\cdots(2n+1)$.

The first tables are of F_n, G_n, F_n', G_n' for $n = 0(1)6$, $x = [0(.05)10; 6D]$, δ^2. In limited ranges where F_n, F_n' have small significant numbers and G_n, G_n' are too irregular to be interpolated, 6D values of $\phi_n, \bar{\phi}_n, \psi_n, \bar{\psi}_n$ are also given, where

$$
\begin{aligned}
\phi_n &= \log f_n(x) - \log\,(2n+1)!! \\
\bar{\phi}_n &= \log \bar{f}_n(x) + \log\,[(n+1)/(2n+1)!!] \\
\psi_n &= \log g_n(x) + \log\,(2n-1)!! \\
\bar{\psi}_n &= \log \bar{g}_n(x) + \log\,[n(2n-1)!!].
\end{aligned}
$$

Recurrence formulas may be used for $n \geqslant 7$. Earlier tables of this kind, by Doodson and others [*MTAC*, v. 1, p. 233], were at interval never less than .1.

The above mentioned tables occupy 53 pages. Then come 6D tables of values of C_n, θ_n, $\bar{C}_n$, $\bar{\theta}_n$, P_n, Q_n, R_n, S_n where

$$G_n + iF_n = C_n \exp(i\{x - \tfrac{1}{2}\pi n + \theta_n\})$$
$$F_n' - iG_n' = \bar{C}_n \exp(i\{x - \tfrac{1}{2}\pi n + \bar{\theta}_n\})$$
$$F_n - iG_n = (-Q_n + iP_n) \exp(i\{x - \tfrac{1}{2}\pi n\})$$
$$G_n' + iF_n' = (R_n - iS_n) \exp(i\{x - \tfrac{1}{2}\pi n\}).$$

These functions are given for $y = 1/x$ in the range 0 to .1 at varying intervals .01, .005, .002, and δ^2.

Finally, there are 6D tables of f_n, $\bar{f}_n$, g_n, $\bar{g}_n$ for $n = 1(1)6$, and $x = [0(.05)m, 6D]$, δ^2, m varying from 2 to 5.35.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **865** (Grubbs).

181.—R. L. Anderson & E. E. Houseman, *Tables of Orthogonal Polynomial Values Extended to* $N = 104$ [*MTAC*, v. 1, p. 148–150].

A recalculation of this table reveals the following complete list of errata

Page	Line	n	*For*	*Read*
615	14	31	broken type	−585
618	22	39	496388	4496388
	14	40	−3583	−2583
625	2	57	+42481	+32481
629	17	61	−2648	+2648
642	38	74	13505	135050
649	14	81	+701925	+701935
653	40	85	−88686	+88686
663	44	95	+2107	−2107
665	45	97	−110308	+110308
669	24	101	−26593	−26592

The last of these was reported by W. F. Brown Jr. in *MTAC*, v. 4, p. 222. The values given for $n = 72$ on p. 640, line 36 are really for $n = 71$. The correct values are: 124392, 966 52584, 15878 63880, 39 89066 92520, 3436 29622 27080.

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182.—N. G. W. H. Beeger, "On the congruence $(p - 1)! \equiv -1 \pmod{p^2}$," *Messenger Math.*, v. 51, p. 149–150, 1922.

On p. 150, $p = 239$ *for* $w = 147$ *read* 107.

This error was discovered as the result of recalculation and extension of Beeger's table of Wilson's quotient to $p < 1000$ by use of the SEAC.

K. Goldberg

NBSCL

183.—H. T. Davis, *Tables of the Higher Mathematical Functions.* V. 2, Bloomington, 1935.

Table 38 of log E_n, p. 298

n	*for*	*read*
11	6358	6400
37	9184	9084
42	0201	0301
44	3908	3907

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184.—R. E. Greenwood & J. J. Miller, "Zeros of the Hermite polynomials and weights for Gauss' mechanical quadrature formula," Amer. Math. Soc., *Bull.*, v. 54, 1948, p. 765–769 [*MTAC*, v. 3, p. 416].

The above table was compared with our similar 13-place manuscript table. Apart from a few discrepancies of only a unit in the last place, the following errors were found in the weight factors:

page	*n*	*for*	*read*
768	7	$\lambda_3 = 0.000\ 971\ 781\ 258$	$\lambda_3 = 0.000\ 971\ 781\ 245$
	10	$\lambda_4 = 0.001\ 343\ 645\ 77$	$\lambda_4 = 0.001\ 343\ 645\ 75$

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R. Zucker
R. E. Capuano

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185.—Z. Kopal, "A table of the coefficients of the Hermite quadrature formula," *Jn. Math. Phys.*, v. 27, 1948, p. 259–261 [*MTAC*, v. 3, p. 473]. On page 260, corresponding to $n = 18$, the sixth value of p,

for 0.000051 59 *read* 0.000051 80

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186.—M. Kraitchik, "On the divisibility of factorials," *Scripta Math.*, v. 14, 1948, p. 24–26 [*MTAC*, v. 3, p. 357–8].

P. 25, $n = 18$, *for* 108514808571661 *read* 226663·478749547

[The fact that this large "prime" factor of $18! - 1$ is composite was proved by A. Ferrier, letter of 16 Nov. 1950, who showed that (for $N = 108\cdots661$)

$$2^{N-1} \equiv 5107\ 05663\ 62894 \pmod{N}.$$

The problem of factoring N was proposed during a demonstration of the SWAC on 29 Jan. 1951. The factor 226663 was discovered by the SWAC in 25 minutes running time. D. H. L.]

187.—NBSMTP *Table of Natural Logarithms*. V. 1, New York, 1941.

P. 184, $x = 18254$ *for* 9.18213 *read* 9.81213

LINDLEY M. WILSON

NBS

188.—WILHELM MAGNUS & FRITZ OBERHETTINGER, *Anwendung der elliptischen Funktionen in Physik und Technik*. Berlin, 1949 [*MTAC*, v. 4, p. 23].

P. 113, col. 4, last line. *For* 1,26928 *read* 1,29628

T. J. HIGGINS
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189.—A. REIZ, "On the numerical solution of certain types of integral equations," *Arkiv Mat., Astr. Fys.*, v. 29A, no. 29, 1943, 21 p. [*MTAC*, v. 3, p. 26.]

On p. 6, Reiz tabulates $\pm x_i$, p_i and α_i, where x_i are zeros of the Hermite polynomials, p_i are $\pi^{-\frac{1}{2}}\cdot$weight factors, and α_i are weight factors$\cdot\exp(x_i^2)$, for $n = 2(1)9$. The following errors of more than a unit in the last (7th) decimal place were found:

	for	*read*
$n = 3$,	1.323931*6*	1.323931*2*
$n = 4$,	1.24022*44*	1.24022*58*
$n = 5$,	1.18148*77*	1.18148*86*
$n = 6$,	1.33584*89*	1.33584*91*
	0.935580*8*	0.935580*6*
	1.136*8912*	1.136*9083*
$n = 7$,	0.89718*39*	0.89718*46*
	1.1013*979*	1.1013*307*
$n = 8$,	1.9816*821*	1.9816*568*
	0.866*8381*	0.866*7526*
	1.071*8011*	1.071*9301*
$n = 9$,	0.8417*403*	0.8417*527*
	1.04*49691*	1.04*70036*

H. E. SALZER
R. ZUCKER
R. E. CAPUANO

NBSCL

UNPUBLISHED MATHEMATICAL TABLES

[EDITORIAL NOTE. The Mathematical Tables Committee of the Royal Society wishes to announce the establishment of a depository of unpublished mathematical tables in the library of the Royal Society. Lists will be published periodically of the tables which have been accepted. The tables will be available for examination in the library and it is proposed to arrange for photo-copies to be supplied as a reasonable charge to those who desire them.

Communications should be sent to the Assistant Secretary, The Royal Society, Burlington House, Piccadilly, London W.1].

116[C, F].—NBSCL—*Radix table for the computation of logarithms to many places using logarithms of small primes.* Preliminary manuscript awaiting final check in possession of NBSCL and deposited in UMT FILE, 45 leaves, hectographed.

This table, which was calculated at the suggestion of J. B. ROSSER, gives the integers a, b, c, d, e, f, and g and the corresponding quantity

$$N = 2^a 3^b 5^c 7^d 11^e 13^f 17^g$$

for about 2000 values of N ranging from 9.996 to about $1 + 2\cdot 10^{-16}$. Exact values of N are given down to $N = 1.155$, thereafter N is given in the form $1 + v$, where v is given to 5S.

The purpose of this table is to facilitate the computation of logarithms to very many places, making use of the fact that the logarithms of 2, 3, 5, 7, 11 and 17 have been given by H. S. UHLER[1] to 330 decimals. The logarithm of 13 was calculated to 328 decimals, using Uhler's values. The logarithm of 37 is obtained as a by-product.

The method of construction and the method of using the table are described in the introduction. The entire computation and most of the preliminary checking was done by ELIZABETH GODEFROY, under the supervision of H. E. SALZER.

[1] H. S. UHLER, "Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7, and 17." Nat. Acad. Sci. Washington, *Proc.*, v. 26, 1940, p. 205–212; "Natural logarithms of small prime numbers," *Ibid.*, v. 29, 1943, p. 319–325.

117[F].—BALLISTIC RESEARCH LABORATORY, *Table of Fermat's quotients*, 236 leaves, tabulated from punched cards. On deposit in UMT FILE.

Let $p = p_n$ be the n-th prime and let $\epsilon = \epsilon(p_n)$ be the least positive integer x such that p divides either $2^x + 1$ or $2^x - 1$ (in other words $\epsilon(p)$ is the exponent of 4 modulo p). Finally let $r(p)$ and $R(p)$ denote the least non-negative remainders on division of $2^\epsilon - 1$ by p and p^2 respectively. Then the main table gives

$$n,\ p = p_n,\ p^2,\ \epsilon = \epsilon(p),\ f = (p-1)/\epsilon,\ r(p) \text{ and } R(p)$$

for $n = 2(1)\,2860$, that is for $p < 26000$. This table was computed in 1949 on the ENIAC by GEORGE REITWIESNER and HOMÉ MCALLISTER during check periods at the suggestion of D. H. L. and is intended to give information about Fermat's quotient[1]

$$q_2 = q_2(p) = (2^{p-1} - 1)/p.$$

This integer is related to R and f by the congruence

$$q_2 \equiv \tfrac{1}{2} fR(R + 2)/p \pmod{p}.$$

The interesting case of q_2 being divisible by p is thus equivalent to R being 0 or $p - 2$. This occurs only twice in the whole table, at $p = 1093$ and $p = 3511$. These exceptional primes were discovered by MEISSNER (1913) and BEEGER (1922); Beeger's calculations[2] are for $p < 16000$. The main

table has been rearranged in three other ways, by sorting the cards on which the results were punched, as follows

1) In two parts according as $2^e \equiv 1$ or $-1 \pmod{p}$
2) According to values of f
3) According to values of f in each of the cases $2^e \equiv 1$ or $-1 \pmod{p}$.

[1] See L. E. Dickson, *History of the theory of numbers*. V. 1, Washington, 1928, Chapter 4.
[2] N. G. W. H. Beeger, "On a new case of the congruence $2^{p-1} \equiv 1 \pmod{p^2}$," *Messenger Math.*, v. 51, 1922, p. 149–150.
"On the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ and Fermat's last theorem," *Nieuw Archif v. Wiskunde*, s. 2, v. 20, 1939, p. 51–54.

118[F].—A. Gloden, *Tables des solutions des congruences* $X^{2^n} + 1 \equiv 0 \pmod{p}$ $n = 4, 5, 6$; $p < 10^4$. Typewritten manuscript, 8 leaves. Deposited in UMT File.

All solutions $<p/2$ of the congruences

$$x^{16} \equiv -1, \qquad x^{32} \equiv -1, \qquad x^{64} \equiv -1 \pmod{p}$$

are given for those primes $p < 10^4$ for which such congruences are possible.

119[F].—A. Gloden, *Tables de décomposition des nombres premiers* $8k + 1$ *dans l'intervalle* 350000–600000 *en* $a^2 + b^2$ *et* $2c^2 + d^2$ *et tables des solutions des congruences* $l^2 + 1 \equiv 0$ *et* $2m^2 + 1 \equiv 0$ *pour ces mêmes nombres premiers*. Handwritten manuscript of 97 leaves, deposited in UMT File. The original manuscript deposited in the Bibliothèque National de Luxembourg.

This is an extension of the author's UMT **113** [*MTAC*, v. 5, p. 28]. The author plans a further extension to 800000.

120[F].—R. M. Robinson, *Stencils for the solution of systems of linear congruences modulo* 2. Box of 1024 IBM punched cards on deposit in the UMT File. Copies of these stencils may be obtained at cost from the Computation Laboratory, University of California, Berkeley 4, California.

Any system of 9 or fewer linear congruences may be solved modulo 2 by simply superposing and sighting 9 or fewer punched cards selected from the set. This set of stencils is used in the application of a sieve method for the solution in integers of systems of linear equations devised by H. Lewy.

121[K].—D. L. Gilbert, *Levels of significance—a direct table of the two-tail probabilities*. 38 leaves, mimeograph manuscript deposited in the UMT File.

This is a direct tabulation of the areas in the tails of the Student-Fisher distribution, that is, of

$$P = P(|u| \geq t) = 1 - n^{-\frac{1}{2}}(B(\tfrac{1}{2}n, \tfrac{1}{2}))^{-1} \int_{-t}^{t} (1 + u^2/n)^{-(n+1)/2} du.$$

Values of $100P$ are given to 1D for $n = 1(1)35$ and $t = 0(.01)5.99$. There is a supplementary table for $n = 1, 2, 3$ which gives the ranges of values of t to 2D for which $P(u \geqq t) = 0(.1)5.5$. Previously published tables of direct

values (most tables, including the recent ones, give values of t for which $P(|u| \geqq t)$ has selected values) have proceeded by steps of .1 for t, the two principal ones being due to W. S. GOSSETT ("Student") [*Metron*, v. 5, No. 3, 1925, p. 105–108], which in its main part gives values to 4D for $n = 1(1)20$ and $t = 0(.1)6$, and to KARL PEARSON [*Tables for statisticians and biometricians*, Part I, 3rd edition, London, 1930, p. 36]. The present tables were calculated by linear interpolation for $0 \leqq t \leqq 6$ and $1 \leqq n \leqq 20$ from Gossett's 1925 tables; for higher values of t and n, approximate methods given by Gossett were used. [Some spot checking by direct calculation of the corresponding incomplete β-functions for $t = 1(.01)1.09$ with $n = 1$ and 10 gave in 8 cases values differing from Gilbert's by 1 in the third decimal place. For a more critical value $n = 20$, $t = 0.5$ a direct evaluation of the incomplete β integral confirmed Gilbert's value of 0.623. C. C. C.]

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 225 Far West Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

Our contribution under this heading, appearing earlier in this issue, is "The California Digital Computer," by PAUL L. MORTON.

DISCUSSIONS

"Floating Decimal" Calculation on the IBM Card Programmed Electronic Calculator

There is a wide variety of problems occurring in run-of-the-mill computing in which it is extremely helpful to represent numbers x as $x_0 \cdot 10^p$ where x_0 has the same significant digits as x within the capacity of the machine used, $1 \leqslant x_0 < 10$, and p is an integer. Three types we might mention are:

1) Problems in which certain computed quantities have magnitudes difficult to estimate.
2) Problems in which one or more quantities have such a wide range of magnitude that no single fixing of the decimal point will suffice for the entire range.
3) Problems in which the setup time involved in estimating magnitudes is not justified by the saving in machine time made possible by fixed decimal calculation.

For application of the IBM Card Programmed Electronic Calculator to such problems, we have devised a calculator programming of the general-purpose type based on the "floating decimal" representation of numbers. A brief explanation of the way the machine is instructed will help in an understanding of what follows.

The calculating unit (IBM 604) of the Card Programmed Electronic Calculator is, for one step in the calculation, instructed to receive numbers

A and B, to perform on them an operation $f_i(A, B)$, and to leave result C where it can be used for further calculation or can be stored. The numbers A, B, and C may have up to ten digits and a sign. The choice of operations f_i is determined to suit the needs of the group using the machine. The more calculation one can put into one operation, the faster the C.P.E.C. can do the problem. Hence, for a large problem, it is sensible to design the operations f_i with this in mind. A more versatile but slower setup is one in which the same set of operations will suffice for any problem. It will rather obviously include among its operations addition, subtraction, multiplication, and division. What additional operations are included is up to the needs of the group using the machine. The choice of operations is put into effect by wiring of plugboards. Most operations are completely instructed by wiring of the plugboards for the 604, but on occasion, it is useful to alter A or B before they are delivered to the calculator or C as it comes from the calculator. This implies special wiring on the tabulator plugboard.

In our programming of the C.P.E.C. for general-purpose floating decimal calculation, we represent a number x by a ten-digit number X. In the representation $x = x_0 \cdot 10^p$, we carry eight significant digits in x_0 and allow p the range $-50 \leqslant p \leqslant 49$. The first eight digits of X are the digits of x_0. The last two digits of X are those of $p + 50$. The sign of X is that of x_0. Examples are:

x	X
$4.1752139 \cdot 10^4$	4175213954
$-1.9920367 \cdot 10^{-20}$	-1992036730

The set of operations which is available in our setup is as follows:

$$
\begin{aligned}
f_0(A, B) &= A \\
f_1(A, B) &= A + B \\
f_2(A, B) &= A - B \\
f_3(A, B) &= A \cdot B \\
f_4(A, B) &= A/B \\
f_5(A, A) &= \sqrt{A} \\
f_6(A, B) &= |A| \cdot B \\
f_7(A, B) &= |A|/B
\end{aligned}
$$

Corresponding to $i = 3, 4, 5, 6, 7$ there is an operation f'_i such that $f'_i = -f_i$.

If no number is delivered to the calculator to be used as A or B in a calculation, then the number already in that position is used. Because numbers delivered to storage cannot be read from storage for use as A or B immediately, it is often convenient to store the result of a calculation in the A or B position in the calculator before the next instructions are given. For all operations except f_0, we automatically store C in the A and B positions. For f_0, C is stored in the B position, and the unused B in the A position.

Operations 6 and 7 are made possible by subsidiary wiring on the tabulator plugboard which replaces A by $-|A|$. Hence, neither of these operations can make use of holding the previous result within the calculator in A position.

Our floating decimal programming for the calculator relies heavily on the program repeat feature of the 604 used in the C.P.E.C. This feature

allows the machine, given a set of instructions, to repeat it several times in one operation and, if desired, to use one subset of these instructions on one time through a set, a different subset the next time. This use of program repeat will make addition or subtraction take two or three times the normal computing time if A and B are different by several orders of magnitude or if C has a magnitude much smaller than that of A and B. As is to be expected, the square root operation will always be slow. The other operations will always be finished in the minimum time for a single operation.

Although it is rather expensive in selectors on the tabulator plugboard, one can easily instruct the machine to make a reasonable (within a factor of 4) guess at the qth root of a number which then is improved by iteration. The guess at the qth root of $x_0 \cdot 10^p$ is $3 \cdot 10^{\left[\frac{p}{q}+50\right]-50}$ where $[x]$ is the greatest integer $\leqslant x$.

The zero of our "floating decimal" setup cannot conveniently have all its digits zero. An explanation of our addition will demonstrate the origin of our representation. Suppose two numbers are added which differ in absolute value only in the eighth significant digit but which are of opposite sign. Then the digits of the result will lead off with zeros. In order to get the answer back to standard representation, we examine the left-hand digit. If this digit is zero, we shift the digits of the answer one position to the left and decrease the calculated p by one. This shifting is continued until the left-hand digit is non-zero. To enable the machine to stop shifting if all eight digits of the answer are zero, we subtract one in the right-hand digit of the answer each time a shift is made after the first. Thus on our machine,

$$x_0 \cdot 10^p - x_0 \cdot 10^p = -1.1111111 \cdot 10^{p-9}.$$

This shifting device gives an obvious means for converting fixed decimal numbers of eight digits or less to floating decimal numbers. Suppose the number of largest magnitude in a column of figures is $x \cdot 10^p$. We punch $x \cdot 10^p$ into the card with standard "floating decimal" representation and all the other numbers with the same last two digits and the same decimal position. Then transferring these numbers through the 604 will convert them to standard "floating decimal" representation.

One additional operation which is useful only in the printing or punching of final results has been included in our planning. If calculations made on the C.P.E.C. are to be used on other IBM machines, it may be desirable to convert results to the fixed decimal form. This procedure will often replace key punching by reproducing. The board is so arranged that if $A = x_0^A \cdot 10^{p_A}$, $B = x_0^B \cdot 10^{p_B}$, $p_B \geqslant p_A$, and the operations code is 2 and 7, then A will come out on Channel C with exponent p_B and the left-hand $p_B - p_A$ digits will be zeros. The digits of x_0^B do not affect this rounding of x_0^A; thus it is not necessary to save an entire storage unit to carry p_B.

When Channels A and B are transmitted to the 604, the significant digits of Channel A are entered in General Storage 1 and 2, its exponent in Factor Storage 4; the significant digits of Channel B are entered in Factor Storage 1 and 2, and its exponent in Factor Storage 3. The sign entries of FS 1 and 2 and GS 1 and 2 are wired from the channel sign hubs, in normal fashion.

The sign wiring of FS 3 and FS 4 is unusual and important to the operation of the board. Its effect is that these two units always read true figures with negative sign. Two pilot selectors are used. The first is picked up whenever a spread read-in card is at the third brushes. The second is picked up immediately by the impulse from "Transfer and S.P. × Control Plus" hub for counter 2A, which is wired to total on all cycles. FS 3 sign hub is wired to common of one position of the first selector. The transferred hub of that selector is wired to the 10 impulse emitter, and the normal hub is wired to common of one position of the second selector. Transferred of the second selector is wired to the 10 impulse emitter, normal is wired to the sign of Channel B. FS 4 is wired similarly in the other position of each of the two pilot selectors, with the final connection to sign of Channel A.

The significant digits of Channel C are in GS 3 and 4, and the exponent in the two lowest order positions of the 604 counter. Channel C sign is to be taken from GS 3 and 4.

The absolute value device is this: an x-punch is used to instruct the machine; it picks up a pilot selector, which transfers the sign of GS 1 and 2 from Channel A sign hub and connects it to the sign of FS 4. Then GS 1 and 2 contains the negative of the absolute value of A.

The coding that has been used for this board is:

No Instruction	Addition	(f_1)
2	Subtraction	(f_2)
3	Multiplication	(f_3)
4	Division	(f_4)
4 and 5	Square Root	(f_5)
7	Transfer	(f_0).

The effect of instruction 2 is just to substitute $-B$ for B. Therefore we also have:

2 and 3	Negative Multiplication	(f'_3)
2 and 4	Negative Division	(f'_4)
2, 4, and 5	Negative Square Root	(f'_5).

If we use the x-punch for absolute value in addition to operation codes, we have:

x, 2, and 3	(f_6)
x, 2, and 4	(f_7)
x and 3	(f'_6)
x and 4	(f'_7).

Calculate selectors 2, 3, 4, 5, and 7 may be picked up in the ordinary way. Selector 1 is to be picked up whenever either operation code 3 or 4 is present. Selector 6 is to be coupled to selector 5. Selector 8 is to be coupled to selector 4.

On the programming chart, some positions are marked with an asterisk. These are positions where the instruction must be selected according to which operation is being performed. The selection is indicated in the "remarks" column. In that column the symbol 012, for instance, refers to program 01, exit 2.

The significance of the symbols at the program positions is determined by the heading of the column in which they appear. For instance, GS 1 and 2 in the Exit 1 column means General Storage 1 and 2 read-out. The same symbol in the Exit 2 column means General Storage 1 and 2 read-in. The abbreviations R & R and RO refer to the counter. We have used the symbol # to help distinguish between abbreviations for General Storage and those for Group Suppression. Thus we have GS 3 and 4 (General Storage 3 and 4) and GS #4 (Group Suppression number 4). We have written miscellaneous orders (Balance Test for Step Suppression, group suppression controls, etc.) in Exit 3 where possible.

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APPENDICES

Suppression Types for General-Purpose Floating Decimal Programming for C.P.E.C. 604

Types of suppression:	
I	Group suppress #1
II	Group suppress #2
III	Group suppress #3
IV	Group suppress #4
V	Suppress on negative balance.
VI	Suppress on positive balance.
VII	Suppress on non-zero balance.
X	Suppress if selector 5 is normal.
XI	Suppress if selector 5 is up.
XII	Suppress on negative balance; suppress if selector 5 is normal.
XIII	Suppress on negative balance; suppress if selector 1 is up.
XIV	Suppress on negative balance if selector 1 is normal; group suppress #1 if selector 1 is up.
XV	Group suppress #1 if selector 4 is up.
XVI	Suppress on negative balance if selectors 1 and 5 are normal; suppress on positive balance if selector 5 is up; group suppress #1 if selector 1 is up and 5 is down.
XVII	Suppress if selector 4 is up.
XVIII	Group suppress #1; suppress if selector 5 is up.
XIX	Group suppress #1; suppress if selector 4 is up.
XX	Group suppress #4 if selector 5 is normal.
XXI	Group suppress #1; suppress if selector 4 is normal.
XXII	Suppress on zero balance if selectors 3 and 7 are normal; suppress if selector 3 is up and 7 is normal; group suppress #1 if selector 7 is up.
XXIII	Suppress on zero balance; suppress if selector 1 is up.
XXIV	Suppress on negative balance; suppress if selector 7 is up.
XXV	Suppress if selector 5 is up; group suppress #2 if both selector 2 and selector 7 are up.

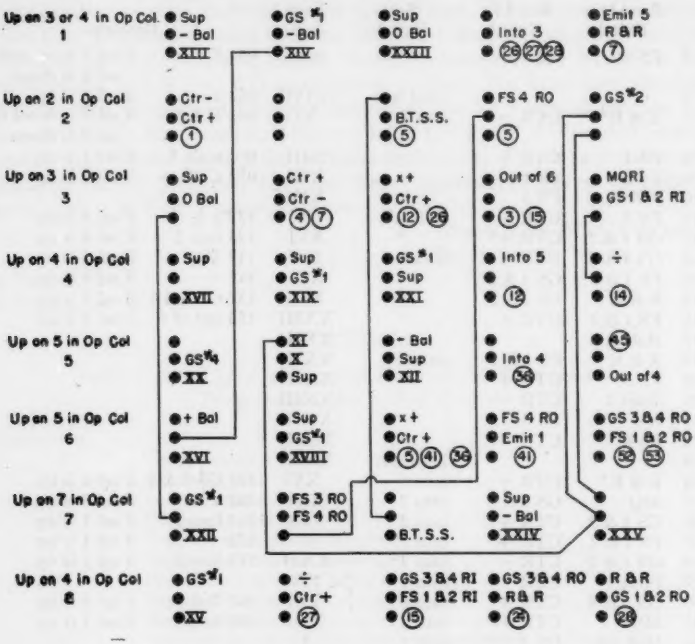
PROGRAMMING CHART

	Read Out Exit 1	Read In Exit 2	Shift Exit 3	Type Sup.	Remarks	
1	FS 1 & 2	CTR +*	GS #4 PU	II	012 CTR −	if sel 2 is up
2	R 0	MQ		XVII		
3	R & R	FS 1 & 2	*	II	033 out of 6	if sel 3 is up
4	FS 3	CTR −*		XVIII	042 CTR +	if sel 3 is up

PROGRAMMING CHART—*Continued*

	Read Out Exit 1	Read In Exit 2	Shift Exit 3	Type Sup.	Remarks	
5	FS 4*	CTR +*	*	none	051 FS 3	if sel 7 is up and sel 2 is down
6			Zero test	XXIII	052 x +	if sel 5 is up
7	R & R*	CTR −*	into 2	XI	053 B.T.S.S.	if sel 7 is down or sel 2 is down
8	FS 4	CTR +		XIII	071 emit 5	if sel 1 is up
9	FS 3	FS 4		XIII	072 CTR +	if sel 3 is up
10	R & R	FS 3		XIV		
11	FS 3	CTR +	B.T.S.S.	X	122 x +	if sel 3 is up
12	GS 1 & 2	CTR +*	*	XVI	123 into 5	if sel 4 is up
13	GS 1 & 2	CTR +	into 4	XII	142 MQ	if sel 3 is up
14	FS 1 & 2	GS 1 & 2*		XIV	142 ÷	if sel 4 is up
15	R & R	FS 1 & 2*	*	XIV	152 GS 3 & 4	if sel 4 is up
16	FS 1 & 2	CTR +		XXIII	153 out of 6	if sel 3 is up
17	Half adj.			XXIII		
18	R & R	FS 1 & 2	out of 2	XXIII		
19	FS 3	CTR +		XXIII		
20	Emit 1	CTR −		XXIII		
21	R 0	FS 3		XXIII		
22	FS 4	CTR −		XXIII		
23			Zero test	XXIII		
24	R & R*	CTR +	into 6	XV	241 GS 3 & 4	if sel 4 is up
25	MQ	GS 3 & 4	into 2	XXI	262 x +	if sel 3 is up
26	GS 1 & 2	CTR +*	into 3*	XIX	263 into 1	if sel 1 is up
27	FS 1 & 2	CTR +*	into 3*	I	272 ÷	if sel 4 is up
28	GS 1 & 2*	CTR −	into 3*	XXII	273 into 1	if sel 1 is up
29	Emit 1	CTR −	into 3	XX		
30	GS 3 & 4	CTR +	into 6	XXI	281 R & R	if sel 4 is up
31	MQ	CTR +	into 2	XXI	283 into 1	if sel 1 is up
32	Half adj.	GS #3 PU	into 2	I		
33	R & R	GS 3 & 4	out of 3	X		
34	Emit 5	MQ	into 3	none		
35	FS 1 & 2	x +	GS #3 DO	X		
36	GS 3 & 4	CTR +*	into 4*	III	362 x +	if sel 5 is up
37	R 0	GS 3 & 4	out of 3	XI	363 into 1	if sel 5 is up
38	Half adj.		into 3	none		
39	R & R	FS 1 & 2	out of 4	none		
40	Emit 5	FS 3	into 2	X		
41	Emit 1*	CTR +*	GS #4 DO	III	411 FS 4	if sel 5 is up
42	Emit 1	CTR −	GS #2 DO	XIX	412 x +	if sel 5 is up
43	FS 3	CTR +		XI		
44	FS 3	x −		X		
45	R & R	FS 3	*	none	453 out of 4	if sel 5 is up
46	FS 1 & 2	CTR −	GS #1 PU	VII		
47	GS 3 & 4	CTR +	GS #1 DO	X		
48	R 0	GS 3 & 4	out of 3	X		
49	R & R		B.T.S.S.	none		
50	MQ	CTR −	into 5	XXV		
51	MQ	CTR −	into 5	XXV		
52	FS 1 & 2*	CTR +		VI	521 GS 3 & 4	if sel 5 is up
53	FS 1 & 2*	CTR −		V	531 GS 3 & 4	if sel 5 is up
54						
55	R & R	GS #3 DO	B.T.S.S.	none		
56		GS #2 PU	Prog. Rpt.	VI		
57	FS 1 & 2	GS 1 & 2		XXIV		
58	FS 3	FS 4		XXIV		
59	FS 3	CTR +		V		
60	FS 1 & 2	GS 3 & 4		V		

DIAGRAM OF THE CALCULATOR SELECTORS



Notes on Numerical Analysis—4
Relaxation and Step-by-Step Methods

Classical methods for the numerical solution of ordinary differential equations are all of the step-by-step type, directly applicable only to equations with conditions specified at one end of the range of integration. The comparatively recent development of relaxation methods provides a new technique for the solution of differential equations for which conditions are given at two distinct points in the range of integration. We shall describe a combination of the two methods which has proved effective in certain cases, for instance, when the behavior of the solution is exponential in one part of the range and oscillatory in another.

To fix ideas consider a second-order equation, from which the first derivative has been removed, of the form

$$(1) \qquad y'' + fy = g,$$

where f and g are functions only of the independent variable x. We can establish a finite-difference equation to replace (1), connecting values of y at three adjacent pivotal points, separated by an interval h, of the form

$$(2) \qquad y_{r+1} + y_{r-1} - (2 - h^2 f_r)y_r - h^2 g_r + \Delta(y_r) = 0.$$

Here, Δ, the difference-correction, is given by

$$\Delta = -\delta^4/12 + \delta^6/90 + \cdots. \tag{3}$$

(We could use a more powerful three term formula, involving a much smaller Δ, but these equations are adequate for our present illustrative purpose.)

An equation of type (2) is to be satisfied at every pivotal point in the range of integration. These equations, moreover, are quite independent of the initial or boundary conditions of the problem, but the latter will determine the method by which such equations are solved.

If the conditions are of the boundary-value type, we shall commonly have specified values y_0 and y_n at the two ends of the range of integration. With initial neglect of Δ, we can therefore solve equations (2) as a set of linear simultaneous equations.

If the conditions are of the initial-value type, we shall be given, or can find fairly easily, two adjacent values y_0 and y_1 at the start of the range. Again, neglecting Δ, we can in this case solve equations (2) by recurrence, obtaining in succession values of $y_2, y_3, \cdots$, etc.

In each case the approximate solution so obtained can be differenced, an approximation to Δ calculated at every point, and with these values inserted as additional terms in equations (2), the respective processes can be repeated to obtain better approximate solutions y. This cycle is carried on until the equations (2) and (3) are accurately satisfied.

Now the linear equations are most easily solved when the coefficients are dominant in the principal diagonal. The use of iteration or relaxation methods is then particularly convenient, and the results are relatively free from error. This dominant diagonal is present whenever the solution is of the exponential type, when f is negative. If the solutions are oscillatory, with f positive, the linear equations are relatively ill-conditioned and their solution much more difficult and less precise.

The recurrence method has precisely the opposite features. In exponential regions there is an uncontrollable growth of error; in oscillatory regions the error is comparatively negligible.

We would therefore like to use relaxation in exponential regions, recurrence in oscillatory regions. In some problems of practical interest the solution has different forms in different regions, and sometimes the given conditions allow us to use the appropriate methods.

Consider, for example, an equation of the form

$$y'' + A(x)y = 0, \qquad y(0) = 0, \tag{4}$$

in which $A(x)$ is large and negative near the origin, later increasing, passing through zero, and having an asymptotic value of say unity. The asymptotic solution, for large x, then has the form

$$y \sim a \sin x + b \cos x,$$

and the problem is to determine the ratio a/b.

The solution begins by increasing rapidly, then decreases and begins to oscillate, as shown in figure 1.

The finite-difference expression for (4) is given by

$$y_{r+1} + y_{r-1} - (2 - h^2 A_r)y_r + \Delta(y_r) = 0. \tag{5}$$

Now the coefficient of y_r is initially greater than 2 in absolute value, and begins to be less than 2 when $A(x)$ passes through its zero. Let the point at which this occurs be called x_c. The solution is exponential for $x < x_c$, oscillatory for $x > x_c$.

Equation (4) is homogeneous in y, so that y can be given any arbitrary value at one particular point other than the origin. Fix the value $y = y_c$ at $x = x_c$. For $x < x_c$ we then have a boundary-value problem easily soluble

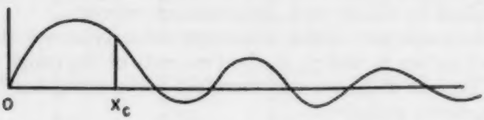


Fig. 1.

by relaxation methods. Let us by this means obtain the first approximation, with Δ neglected, in this range. This solution provides at least two starting values, at x_c and x_{c-h}, from which we can carry forward the solution by recurrence into the oscillatory region.

Since the same equation (5) has been used throughout, the approximation so obtained is "smooth" throughout the entire range. We can therefore difference and calculate Δ everywhere. A better solution can now be obtained in the exponential region by relaxation. The improved value at x_{c-h}, together with the given value at x_c and the difference-corrections, enable us again to carry this better approximation into the oscillatory region, as far as necessary. The process can be repeated until equations (5) are everywhere satisfied.

There is here a nice computational point. The exponential region is completely self-contained, and the final solution, with Δ included, could be obtained before ever proceeding into the oscillatory region. If this were done, however, the accurate exponential solution would not fit well with the first approximate oscillatory solution: there would be some sort of discontinuity at x_c, the differences would not be smooth, and the calculation of Δ uncertain. For this reason the application of Δ is deliberately deferred until it can be effected smoothly over the whole range.

The same technique will help us to throw light on a matter hitherto obscure in relaxation. From a first approximation to the solution of a boundary-value problem the difference correction is not immediately calculable near the ends of the range, since central differences do not exist there.

The first approximation, however, enables us to extend this approximate solution, by recurrence methods, outside the chosen range in both directions (provided, of course, that the function exists outside this range). We can then compute Δ not only inside, but also at points outside the range. The internal values enable us to improve our internal solution, and these new internal values, together with the external Δ's, allow us to obtain a new solution externally, to the same degree of approximation as the new internal values. The process is repeated until there is no further correction. The number of values determined outside the range depends, of course, on the order of significant differences in Δ.

If the solution is everywhere of exponential type, the external recurrence may not be very accurate, the error at a point growing with its distance from the particular end point. Since these values are required, however, only for the computation of Δ, which has small coefficients multiplying the differences, some error can be tolerated.

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1. Anon., "Digital computer," *Radio Electronics*, v. 21, June 1950, p. 9.

A brief description of General Electric's OMIBAC (Ordinal Memory Inspecting Binary Automatic Calculator), a binary machine having a magnetic cylinder memory particularly useful for repetitive problems.

2. E. C. Berkeley, "Simple Simon," *Scientific American*, v. 183, Nov. 1950, p. 40–43, illustrs.

The author describes a simple digital relay computer called Simon which was constructed to illustrate the principles of automatic digital computers [*MTAC*, v. 4, p. 239].

M. M. Andrew

NBSMDL

3. Edmund C. Berkeley, "The relations between symbolic logic and large-scale calculating machines," *Science*, v. 112, p. 395–399.

This is an elementary discussion of some of the logical problems arising in constructing and programming for a modern automatic computer. A representative machine, capable of certain simple logical and arithmetical operations is described, and the "program" for calculating square roots on it is discussed in some detail.

Michael Montalbano

NBSMDL

4. J. L. Hill, *Design features of a magnetic drum information storage system*, Engineering Research Associates, Inc., St. Paul 4, Minnesota, 1950, 8 p. (Transcript of a paper delivered at the Association for Computing Machinery Conference, March 28–29, 1950, Rutgers University), figs.

An expository article (for a supplementary description of this information storage system see *MTAC*, v. 4, 1950, p. 31–39).

5. Cuthbert C. Hurd, *The Selective Sequence Electronic Calculator of the International Business Machines Corporation*, Carbide and Carbon Chemicals Corporation, Engineering Development Division, Theoretical Analysis Department, Report No. K-134, Feb. 4, 1948, 6 pages, diag.

An expository paper.

6. Sherman C. Lowell, *Digital Computer Research at the T. R. E., Malvern*, Office of Naval Research, London, Technical Report ONRL—35–50, Mar. 29, 1950, 3 p.

The main design features of the Telecommunications Research Establishment computer are described. It is a general-purpose, high-speed, parallel, and binary machine.

7. Sherman C. Lowell, *The Manchester University Digital Computer*, Office of Naval Research, London, Technical Report ONRL—34–50, Mar. 29, 1950, 4 p.

This is an expository article which describes a high-speed, serial, and binary machine and its applications and also a larger machine of similar design being built by Ferranti Brothers. The Ferranti computer will be superior from the engineering and maintenance standpoint.

8. Office of Naval Research, *Digital Computer Newsletter*, v. 2, no. 4, December 1950, 5 p.

The present status of the following digital computer projects is treated briefly in this number.

1. NRL Computer (NAREC)
2. Moore School Automatic Computer (MSAC)
3. Naval Proving Ground Calculators
4. SEAC (National Bureau of Standards Eastern Automatic Computer)
5. Aberdeen Proving Ground Computers:
   Coupling of IBM Relay Calculators
   The ENIAC
6. SWAC (National Bureau of Standards Western Automatic Computer)
7. The Institute for Advanced Study Computer
8. Project Whirlwind
9. The ACE Pilot Model, Teddington, England
10. Data Handling and Conversion Equipment:
    Digital Reader
    Digital Data Recording System
    Analog Digital Converter.

News

American Statistical Association.—On Thursday December 28, 1950, at a meeting of the Association in Chicago, a round table discussion was held which had as its topic, "What can high-speed electronic computing equipment do for and to statistics?" The moderator was William G. Madow, University of Illinois. Mr. S. N. Alexander, NBS, discussed the subject from the viewpoint of an electronic engineer while Mr. B. Schreiner, A. C. Nielsen Company, talked about the problem as seen by an expert user. A discussion followed by H. C. Grieves, Bureau of the Census, and John J. Finelli, Metropolitan Life Insurance Co. Organizations joining in the meeting were the Psychometric Society, the Institute of Mathematical Statistics, and the Association for Computing Machinery.

Centre National de la Recherche Scientifique.—An international colloquium on "Modern calculating machines and the human mind" was held at the CNRS on January 8 through 13th at Paris. The program for the meeting was as follows:

First Session, *Recent Technical Progress in Large-Scale Calculating Machines*	LOUIS DE BROGLIE, Institut Blaise Pascal, Secretary of Academy of Sciences, *Chairman*
"The Mark II, III, IV Machines"	H. H. AIKEN, Director of the Computation Laboratory, Harvard University
"A magnetic automatic calculating machine"	A. D. BOOTH, Birkbeck College, University of London
"Development of electronic digital computers at the National Bureau of Standards"	E. W. CANNON, National Bureau of Standards, Washington, D. C.
"Research activities of the National Physical Laboratory"	E. M. COLEBROOK, National Physical Laboratory, Teddington, England
"The Institut Blaise Pascal Machine"	L. COUFFIGNAL, Director of the Computer Laboratory, Institut Blaise Pascal
"Mathematical machines in Sweden"	STIG EKELOFF, Chalmers Tekniska Högskola, Goteborg
"Research in progress at the University of Brussels"	P. GERMAIN, Université Libre, Brussels
"Short-cut multiplication in a parallel-decimal automatic calculating machine"	E. J. PETHERICK, S. H. HOLLINGDALE, Royal Aircraft Establishment, Farnborough, Hants, England
Second Session, *Applications of Mathematical and Scientific Problems to Large-Scale Calculating Machines*	M. CAQUOT, Institut Blaise Pascal, *Chairman*
"Numerical integration of the wave equation"	F. H. VAN DEN DUNGEN, University of Brussels
"Explanation of numerical methods of integration of systems of linear partial differential equations and the results obtained"	M. PICONE, University of Rome
"Different problems a machine can solve"	A. M. UTTLEY, Telecommunications Research Establishment, Worcester, England
"Les erreurs de chute dans les calculs systematiques"	A. VAN WIJNGAARDEN, Mathematische Centrum, Amsterdam
"Operational experience with the EDSAC"	M. V. WILKES, Director, University Mathematical Laboratory, Cambridge
Third Session, *Large-Scale Calculating Machinery and the Logic and Physiology of the Nervous System*	L. LAPICQUE, Institut Blaise Pascal, *Chairman*
"A presentation of the homeostat"	W. R. ASHBY, Department of Research, Barnhouse, England
"Some new analogies between the structure of computers and the structure of the brain"	L. COUFFIGNAL, Institut Blaise Pascal, Director, Laboratoire de Calcul Mécanique
"Large-scale calculating machines and the physiology of the nervous system"	H. GASTAUT, Faculté de Médecine de Marseille
"Mechanical realization of models of the brain structure; presentation of artificial animals"	W. GREY WALTER, Burden Neurological Institute
"Work in Sweden"	G. KJELLBÖRG, Tekniska Högskolan, Stockholm

Discussion	R. LORENTE DE NO, Laboratories of the Rockefeller Institute
Discussion	W. MCCULLOCH
"Developments in automatic computing machinery in the Spanish school"	P. PUIG-ADAM, Academy of Sciences, Madrid
"Présentation des appareils de Leonardo Torrès-Quevedo: 1. le joueur d'échecs automatique 2. le télékine, premier appareil construit pour le radio-guidage des bateaux 3. les fusées logarithmiques de la machine à résoudre les équations logarithmiques"	G. TORRÈS-QUEVEDO, Ingénieur des Ponts et Chaussées de Madrid

Panel on Electron Tubes.—The Conference on Electron Tubes for Computers was held in Atlantic City on December 11th and 12th, 1950, under the joint sponsorship of the AIEE and the IRE in collaboration with the Panel. The program was as follows:

Computer Experience with Electron Tubes	MINA REES, ONR, *Chairman*
"Review of AIEE conference on electron tubes for instrumentation and industrial use"	W. R. CLARK, Leeds and Northrup Co.
"Experience with receiving-type vacuum tubes on the Whirlwind Computer Project"	E. S. RICH, MIT
"Electron tube experience at IBM"	J. A. GOETZ and A. W. BROOKE, IBM
"Performance of electron tubes in the ENIAC"	WRIGHT E. ERION and HOMER W. SPENCE, Aberdeen Proving Ground, Maryland
Electron Tube Problems	S. N. ALEXANDER, NBS, *Chairman*
"Design and operation of tubes for long life"	J. O. MCNALLY, BTL
"The JETEC approach to the tube reliability problem"	J. R. STEEN, Sylvania Electric Products, Inc.
"Cathode inter-face impedance and its effects in aged vacuum tubes"	H. B. FROST, MIT
"Cathode impedance and tube failure"	L. S. NERGAARD, RCA
"6SN7WGE and 6AN5; mutual conductance dispersion and the effect of low duty cycle operation on long life performance"	I. LEVY, Raytheon Manufacturing Co.
"Open discussion in improved electron tubes for computers"	
Special Purpose Computer Tubes	A. L. SAMUEL, IBM, *Chairman*
"A stable binary electrostatic storage system"	A. M. CLOGSTON, BTL
"Recent experiences with the selective electrostatic memory tube"	J. RAJCHMAN, RCA
"The MIT storage tube"	P. YOUTZ, MIT
"The development of the Rogers Additron"	T. VAN DYK, Rogers Majestic, LTD.
"A proposal for a binary adder tube utilizing beam deflection principles"	D. H. GRIDLEY, NRL
"Special cold cathode discharge tube for counting and switching applications"	M. W. WALLACE and J. HENEY, Federal Telecommunication Laboratories

"A decimal counting tube"	T. R. KOHLER, Philips Laboratories
"New improvements and applications in Remtron counter tube"	FRANK J. COOKE, Remington Rand Corp.
Tube Manufacture and Crystal Diode Experience	J. G. BRAINERD, University of Pennsylvania, *Chairman*
"Design and manufacture of electron tubes for electronic computer service"	R. E. HIGGS and H. E. STUMMAN, RCA
"Problems in the manufacture of special tubes for computer usage"	R. L. MCCORMACK, Raytheon Manufacturing Co.
"Development of the 7AK7"	R. W. SLINKMAN, Sylvania Electric Products, Inc.
"Some problems involved in the manufacture of germanium diodes"	N. DEWOLFF, General Electric Co.
"Experience with germanium diodes in the SEAC program"	H. WRIGHT, NBS
"Crystal diode life experience in the Whirlwind Computer circuits"	H. B. FROST, MIT
Williams Type Storage	J. H. BIGELOW, Institute for Advanced Study, *Chairman*
"Tube experience in the SWAC"	H. D. HUSKEY, NBS
"The SEAC memory using Williams storage"	W. W. DAVIS, NBS
"The selection of cathode-ray tubes for Williams storage"	J. H. POMERENE, Institute for Advanced Study
"Methods of testing cathode-ray tubes for service in Williams storage systems"	D. FRIEDMAN, NBS
"Theory of storage in cathode-ray tubes"	J. KATES, University of Toronto
"Progress report on the electron mechanism on the Williams storage technique"	A. W. HOLT, NBS
"Space charge effects in Williams storage tube"	L. BRILLOUIN, IBM

OTHER AIDS TO COMPUTATION

A New Differentiating Machine

The first step in the design of a differentiating machine is the selection of a mechanical analogue for the derivative of a plotted function. The incorporation of this mechanical differential ratio into a machine that will draw the derived curve of a plotted function will follow with the use of the proper linkages, gears, etc., that will transfer the relative value of this mechanical $\frac{dy}{dx}$ to a writing pen for tracing the derived curve.

The most common type of differentiating machine that has been built uses a tangent line analogue based on the geometric concept of the derivative. The differentiating machine built in 1904 by J. E. MURRAY[1] employs this tangent line analogue in the form of two dots on a celluloid plate. So long as the dots remain on the curve to be differentiated, the chord connecting them is approximately parallel to the tangent to the curve at the mid point between the dots, then by a system of connecting linkages, a writing pen is caused to draw the derived curve.

ARMIN ELMENDORF's differentiating machine[2] employs a tangent line analogue in the form of a sharp edged wheel which is rolled along the curve to be differentiated. A pulley system of parallel linkages connects the wheel to a writing pen. Both Murray's and Elmendorf's machines can be adapted to use a "normal mirror" for finding the slope of a curve. A line perpendicular to a mirror held normal to a curve will be a tangent line.

F. E. MYARD[3] based his machine on a different analogue from those already mentioned. His machine develops a differential ratio when a single point is made to trace a curve. This is done by an adding unit (a set of differential gears) and a crown wheel (integrating wheel) which together form the differential ratio $\frac{dy}{dx}$. This ratio is plotted as a curve.

No attempt is made in this article to evaluate the above mentioned machines as differentiating devices.

A new differentiator has been designed and built by CYRIL P. ATKINSON (Fig. 1). It is a "tangent line analogue" type of differentiating machine

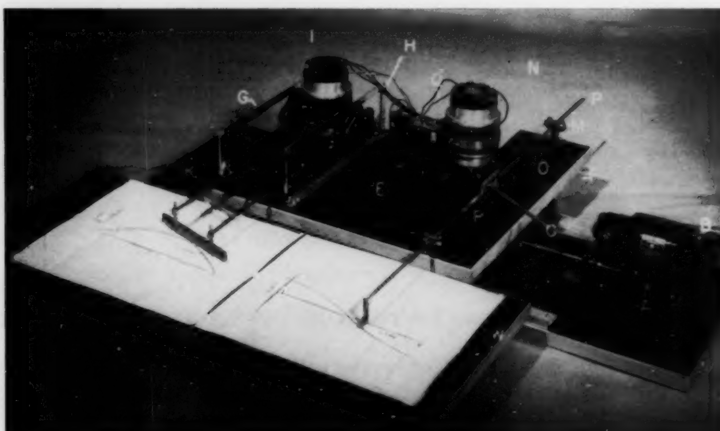

FIG. 1. Atkinson's differentiator.

using a "tracer bar" to measure the slope of the tangent line. The relative value of the tangent of the angle of slope of the tracer bar is transferred to the writing pen by a pair of synchro-transmitters (selsyn motors) in such a way that the motor attached to the writing pen will cause it to describe the derived curve to a scale determined by the constants of the system.

It had been hoped that the differentiator could be operated continuously in the differentiation of a curve and thus produce a continuous derivative. This plan had to be abandoned since an operator is unable to move the tracer bar with such finesse to cause the writing pen to draw a smooth curve. In the process of differentiation the machine is moved from interval to interval

across the original curve by means of the motor and drive shaft. At each interval a slope is determined by the tracer bar and a point is tapped by the writing pen, so that a series of points describes the derivative. A smooth curve drawn through these points is the derived curve. The units of the derived curve can be obtained from the scale factor of the machine. If the units of the abscissa of the original curve are equal to the distance between the centerline of the writing pen M of Fig. 1 and the center of rotation of the shaft of selsyn motor N, the units of the ordinates of the derived curve are equal to the ordinate units of the original curve divided by the units of the abscissa of the original curve. The abscissa units are the same for both curves.

Various curves have been differentiated by the machine and the accuracy was determined to be from 2.5 to 5%. Fig. 1 shows the cosine curve under the writing pen of the differentiator. This cosine curve was produced by differentiating the sine curve shown under the tracer bar. Harmonic curves can be differentiated most easily when they are plotted with the units of the abscissa equal to distance between the centerline of the writing pen and the selsyn motor. When the slope of the tracer bar must exceed 45 degrees to be tangent to a curve as in the differentiation of a circular arc of 90 degrees, a technique that is employed is to determine the normal to the curve by holding the tracer bar across the curve. By observing the reflection of the curve in the polished surface of the tracer bar and causing the reflection to be continuous with the original curve a very accurate normal can be obtained. Since a plot of the slope of the normal would be a curve of $-\frac{dx}{dy}$, it is only necessary to take the negative reciprocal of this curve to obtain the derivative.

Some of the defects of the earlier differentiators were kept in mind in the development of the present differentiator. For example, the tracer bar, the counterpart of which was obscured in both Murray's and Elmendorf's machines, is placed immediately before the operator and can be viewed from above and held tangent or normal to the curve without interference from other parts of the mechanism. Also, the writing pen, which is driven by a synchro-transmitter, can be placed in any position the designer desires, even in an adjoining room if need be. It was placed so the operator would have an unimpeded view of his work, both in the differentiation of the original curve and in the plotting of the derivative.

The defects of the differentiator are (1) the point by point method of producing the derived curve; (2) the inaccuracies due to (a) the approximation of the derivative by a chord, (b) selsyn motor inaccuracies, (c) mechanical inaccuracies.

The machine could be improved by increased precision in the dimensions of the moving parts. The use of larger and more accurate selsyn motors would allow a derived curve to be drawn to a larger scale. The incorporation of photo tubes into the "tracer bar" to replace the points which are used to trace the original curve would make the machine automatic and cause it to draw a continuous derived curve.

The comparison of the present machine with the machines mentioned earlier in this paper as to cost, performance, etc., and a more detailed account of the theory of differentiating machines awaits a more general article.

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[1] J. ERSKINE MURRAY, "A differentiating machine," R. Soc. Edinburgh, *Proc.*, v. 25, pt. 1, 1904, p. 277–280. (Reprinted in E. M. HORSBURGH, *Modern Methods and Instruments of Calculation*. London and Edinburgh, 1914, p. 217–219.)

[2] ARMIN ELMENDORF, "Mechanical differentiation," Franklin Institute, *Jn.*, v. 185, 1918, p. 119–130.

[3] F. E. MYARD, "Nouvelles solutions de calcul grapho-mechanique—derivographes et planimeters," *Le Genie Civil*, v. 104, 1934, p. 103–106.

[4] MEYER ZUR CAPELLEN, *Mathematische Instrumente*. Ann Arbor, 1947 (originally Leipzig, 1944).

[5] J. LIPKA, *Graphical and Mechanical Computations*. New York, 1918.

[6] F. J. MURRAY, *The Theory of Mathematical Machines*. 2nd rev. ed., New York, 1948.

BIBLIOGRAPHY Z-XV

9. H. L. ANDREWS, "Nomogram for G-M counter resolving time corrections," *Rev. Sci. Instruments*, v. 21, 1950, p. 191.

10. J. G. BAYLY, "An analog computer," *Rev. Sci. Instruments*, v. 21, 1950, p. 228–231.

A device is described for solving the equation $\frac{dx}{dt} = -\lambda x + f(t)$, where t is real time. The unknown function x is the rotation of a shaft driven by a servomotor whose rate of rotation $\frac{dx}{dt}$ is regulated to equal a voltage $-\lambda x + f(t)$. $\frac{dx}{dt}$ is measured by a tachometer and the difference between it and $-\lambda x + f(t)$ is used as an input for a servo-system of the type with a "memory circuit" described by WILLIAMS & UTTLEY.[1] A helical potentiometer on the output shaft yields the voltage λx and $f(t)$ is introduced by a manual follower. Various applications to atomic pile calculations are described.

F. J. M.

[1] F. C. WILLIAMS & A. M. UTTLEY, "The velodyne," Inst. Elec. Engrs., *Jn.*, v. 93, part IIIA, 1946, p. 1256–1274.

11. A. BEISER, "A slide rule for nuclear emulsion calculations," *Rev. Sci. Instruments*, v. 21, 1950, p. 933–934.

12. L. M. HAUPT, "Solution of simultaneous equations through use of the A.C. network calculator," *Rev. Sci. Instruments*, v. 21, 1950, p. 683–686.

The system of equations to be solved are realized by means of Kirchhoff's laws. The coefficients are represented by impedances either directly or by reflection through one-to-one transformers. This last device is used to remove the restrictions which normally apply to real transfer impedances.

F. J. M.

13. W. A. McCool, "Frequency analysis by electronic analog methods," NRL *Report* no. 3724, Aug. 21, 1950.

The complex Fourier transform $F(\omega)$ of the real function $f(\tau)$ is expressed as

$$F(\omega) = \int_{-\infty}^{+\infty} f(\tau)e^{-i\omega\tau}d\tau.$$

$F(\omega)$ may be evaluated on analog equipment by a solution of the linear integrodifferential equation

$$y' + \omega^2 \int ydt = f(t); \qquad y(0) = \int_0 ydt = 0.$$

From a solution of a similar system of equations one may obtain the real inverse Fourier transform of $F(\omega)$. An illustrative example and an analog machine program are given. The author has reported several omissions and errata that do not affect the results.

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14. Richard McFee, "A trigonometric computer with electrocardiographic applications," *Rev. Sci. Instruments*, v. 21, 1950, p. 420–426.

The author points out the usefulness of a double electronic resolver which will give the three spherical coordinates of a vector from its three cartesian coordinates expressed as voltages. Such a spherical resolver is readily obtained by combining two ordinary resolvers of the two dimensional type. An electronic resolver which yields $(e_1^2 + e_2^2)^{\frac{1}{2}}$ and the arctan e_2/e_1 is described. The input voltages e_1 and e_2 are used to modulate two sinusoidal voltages of the same frequency, 90° out of phase. These modulated voltages are added and the amplitude is detected to yield $(e_1^2 + e_2^2)^{\frac{1}{2}}$. The phase of this sum voltage is also detected by a pulse technique. A pulse is emitted when the reference voltage passes through zero and also one when the sum voltage passes through zero. The difference in time between these is the required angle arctan e_2/e_1. Special provision is made to yield a zero angle for zero amplitude and for obtaining the base line for amplitude measurements. A proposal is also made for displaying the results in the three dimensional case.

F. J. M.

15. F. G. Fender, "Aerocom. . . . An analog computer," *Northwestern Engineer*, v. 9, no. 1, Mar. 1950, p. 10–12, 32.

This article describes a large analog computer located in the Aerial Measurements Laboratory at Northwestern Technological Institute. This device "now consists of the following types of equipment: inductors, resistors, and capacitors; special amplifiers; special synchronous switches and relays to provide repetition of the transients and to re-establish initial conditions between such repetitions; various function generators; single and dual gun cathode ray oscilloscopes to provide visual representation of out-

puts together with cameras for making permanent records; and various standard electrical devices for calibration and testing. Circuit connections are made by busses with selector switches and by plugs and jacks. . . ." The cost was about $250,000.

F. J. M.

16. R. L. GARWIN, "A differential analyzer for the Schrödinger equation," *Rev. Sci. Instruments*, v. 21, 1950, p. 411–416.

The equation $\varphi'' - [E - V(t)]\varphi = 0$ is solved by means of feed back amplifier integrators. The potential $V(t)$ is obtained by a manual follower. The design considerations for a simple and relatively inexpensive instrument are given at some length and also its use for the solution of the appropriate characteristic value problems.

F. J. M.

17. B. O. MARSHALL, JR., "The electronic isograph for roots of polynomials," *Jn. Appl. Phys.*, v. 21, 1950, p. 307–312.

A polynomial $f(z) = \sum_{j=1}^{n} a_j z^j$ with real a_j can be represented as a function of R and θ when $z = R \exp i\theta$, i.e., $f(x) = \sum_{j=1}^{n} a_j R^j \cos j\theta + i \sum_{j=1}^{n} a_j R^j \sin j\theta$. If R is entered manually as a shaft rotation, it is possible to obtain d.c. voltages corresponding to the quantities $a_j R^j$ by a suitable arrangement of potentiometers. These d.c. potentials are applied to commutators to give a square wave voltage of frequency $60j$ per second. All but the fundamental of this voltage is filtered out and added to yield the real and imaginary parts of $f(z)$. These are used as the horizontal and vertical deflection voltages of an oscilloscope. R is varied until a zero is observed. The instrument handles polynomials of the tenth degree and the output voltages have three figure accuracy under suitable scaling.

F. J. M.

18. FRANÇOIS-HENRI RAYMOND, "Sur un type général de machines mathématiques algébriques," *Ann. Telecommun.*, v. 5, 1950, p. 2–19.

The author considers three particular mathematical machines of the continuous type. One is a machine for solving systems of linear algebraic equations and associated problems (e.g., inversion of a matrix, characteristic values of a matrix). This will be called the "algebraic machine." The other two machines are for solving systems of differential equations. One machine is capable of handling larger systems than the other. However, the theory of both is the same and either machine will be referred to as a "differential analyzer." After some introductory remarks the author briefly describes the operation and applications of the machines. The next portion of the paper, which is of greatest interest to mathematicians, is a discussion of the stability and precision of the devices. These phases will be elucidated below.

Consider the system of n linear equations (in matrix form) $Ax = b$ where $A = \|a_{ij}\|$ is a square matrix and x and b are column vectors. It is assumed

that by a preliminary manipulation all the coefficients of A have been reduced (by a scale factor) such that $|a_{ij}| \leqq 1$. It is also assumed that to each coefficient a_{ij} there is attached an error α_{ij} whose upper bound is a certain percentage of a_{ij}. Similarly for the b_i (components of the vector b). Hence the machine actually solves the equation $(A + \alpha)x' = b''$. The problem is to estimate $|x_i' - x_i|$, that is the difference between the observed solutions x_i' and the actual solutions x_i. By using the REDHEFFER formula[1] the author obtains the following result:

$$\Delta \leqq \epsilon D^{-1}(L + n^{\frac{1}{2}}X_M)(T/(n-1))^{(n-1)/2}$$

where $\Delta = \max_i |x_i' - x_i|$, ϵ = percentage of the coefficient a_{ij} which forms the error α_{ij}; i.e., $\alpha_{ij} = \epsilon a_{ij}$ and $|b_i'' - b_i| = \epsilon |b_i|$, D = the modulus of the determinant of A, T = the trace of AA_t (A_t = transpose of A), L = the length of the vector $b = [\sum_{i=1}^{n} b_i^2]^{\frac{1}{2}}$, $X_M = \max_i |x_i'|$, n = number of equations in the system.

The quantity Δ measures the percentage error. However it appears to the reviewer that the scaling operations are not considered in sufficient detail; for while the percentage error may be small, the actual error may be large if a large scale factor has been used. Also the justification for the assumption that $\alpha_{ij} = \epsilon a_{ij}$ is not obvious. Noise is not considered.

The algebraic machine is essentially of the GOLDBERG-BROWN type[2]; and the stability of the system depends on the characteristic roots of the matrix A. In writing the equilibrium equations for the machine, the following matrix form is obtained:

$$[A - ((n+1)/G)I]x - b = 0 \tag{1}$$

where $G(p)$ is the gain characteristic (in operational form) of the amplifiers used, ($p = d/dt$ is the Heaviside operator). If μ_ν, $\nu = 1, 2, \cdots, n$ are the characteristic roots of A, then $\mu_\nu = (n+1)/G(p)$. If the amplifiers are so constructed that $\mathfrak{R}G(p) < 0$ when $\mathfrak{R}p > 0$ then $\mathfrak{R}p > 0$ if $\mathfrak{R}\mu_\nu > 0$ and the machine is stable. [$\mathfrak{R}$ = "real part of."]

In the differential analyzer the equations to be solved are (in matrix form)

$$g(t) + [Ap^3 + Bp^2 + Cp + D]Y = 0 \tag{2}$$

where A, B, C, and D are square matrices with constant coefficients, $g(t)$ and Y are column vectors and $p = d/dt$. In a manner similar to that used in deriving (1),

$$\{[A - ((n+1)/G)I] + B\alpha + C\alpha^2 + D\alpha^3\}V + g = 0$$

is obtained. [In the ideal case, $G = \infty$ (infinite gain amplifier) and $\alpha(p) = -p^{-1}$.] As in the algebraic case, everything depends on the characteristic roots of the matrix

$$A + B\alpha + C\alpha^2 + D\alpha^3. \tag{3}$$

In the ideal case, the characteristic frequencies are the roots of the determinantal equation

$$\det|A - Bp^{-1} - Cp^{-2} - Dp^{-3}| = 0.$$

Call them p_k. The characteristic roots of the matrix (3) will differ from the p_k by small amounts, δp_k which the author computes later (using the first few terms of a Taylor series expansion). However this matrix (3) may have extraneous roots, q_ν, $\nu = 1, 2, \cdots$.

Under the assumptions $|G(q)| < |G(0)|$ and $|\alpha(p)|$ is small ($\sim 10^{-2}$) it can be shown that the q_ν depend only on the matrix A and not on B. The author states that if this is not the case his results are not valid and that a general discussion appears difficult and is outside the scope of the present study.

Making certain (practical) assumptions regarding the physical system, for example, assuming $G(p) = -G_0/(1 + pT_0)$ where $G_0 = |G(0)|$ and also assuming (for the present) that A is symmetric, the author obtains the results that two sufficient conditions for the correct functioning of the differential analyzer are (a) $|p_k| < 2\pi\Delta f_0$, (b) $|q_m| > 2\pi\Delta f_0$ where $|q_m| = \min_\nu |q_\nu|$ and Δf_0 is the band-width of the amplifier ($2\pi\Delta f_0 T_0 = 1$). Using the Redheffer formula again, it is also shown that

$$q_m \geqq -[1 + G_0/(n+1)\|A\|((n-1)/T)^{n-1}]T_0^{-1}$$

where $\|A\| = |\det A|$ and T is the trace of A.

The author remarks that (a) can always be satisfied by an appropriate change of scale of the independent variable in the original system of equations. Since G_0 is always large ($\sim 10^4$) in a well designed amplifier, he concludes that the critical parameter, from the point of view of stability and precision, is $\|A\|/T^{n-1}$. Errors in the matrices A, B, C, D are not considered (neither steady state nor random).

Certain results are obtained if the condition that the matrix A be symmetric is relaxed.

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[1] R. Redheffer, "Errors in simultaneous linear equations," *Quart. Appl. Math.*, v. 6, 1948, p. 342–343.

[2] E. A. Goldberg & G. W. Brown, "An electronic simultaneous equation solver," *Jn. Appl. Physics*, v. 19, 1948, p. 339–345 [*MTAC*, v. 3, p. 329–330].

19. A. H. Scott, "An instrument for mechanically differentiating curves," *Rev. Sci. Instruments*, v. 21, 1950, p. 397–398.

A Coradi intergraph is modified by replacing the pen by a stand holding a slide rule glass slide. The method of operation is carefully described in this note. An accuracy of two percent in the permissible range of slopes was obtained.

F. J. M.

20. H. R. Seiwell, "A new mechanical autocorrelator," *Rev. Sci. Instruments*, v. 21, 1950, p. 481–484.

The device evaluates

$$\int_a^b f_1(t)f_2(t)dt$$

by means of two ball cage integrators. f_1 and f_2 are entered by manual followers from graphs. f_2 is utilized as the displacement of the cage of the first integrator whose disk is driven at a constant rate. The output of the first integrator drives the disk of the second and the cage displacement of the second integrator is proportional to $f_1(t)$. The output of the second integrator is registered on a counter and is the required answer when the interval of integration has been covered. The accuracy of the device depends on the slope of the functions f.

F. J. M.

21. H. Shimizu, P. J. Elsey & D. McLachlan Jr., "A machine for synthesizing two-dimensional Fourier series in the determination of crystal structures," *Rev. Sci. Instruments*, v. 21, 1950, p. 779–783.

The stators of a pair of selsyns are connected so that a voltage across the rotor of the first induces a voltage in the rotor of the second proportional to the cosine of the sum of the angles of rotation of each rotor from a reference position. Thus if the first is rotated an amount nx and the second an amount my a term $a_{nm} \cos (nx + my)$ can be represented. These voltages can be immediately added to produce $F(x, y) = \sum a_{nm} \cos (nx + my)$. The coordinates x and y are entered as the rotation of two shafts and the rotations nx and my are taken from these by a suitable arrangement of gears. In the present device n and m each have a range of 8 but it is planned to extend this range to 16.

F. J. M.

22. F. C. Snowden & H. T. Page, "An electronic circuit which extracts antilogarithms directly," *Rev. Sci. Instruments*, v. 21, 1950, p. 179–181.

This circuit is based on the use of a 6SK7-GT as an inverted triode. The grid current is then proportional to the antilogarithm of the applied plate voltage for a range of 15 volts and for a range of grid current from .33 to 1 ma.

F. J. M.

23. Robert R. Reid & Du Ray E. Stromback, "Mechanical computing mechanisms," *Product Engineering*, v. 20, 1949, August no. 8, p. 131–135, Sept. no. 9, p. 119–123, Oct. no. 10, p. 126–130, Nov. no. 11, p. 121–124.

In the first article of this series of four, specifications for the input and output of mechanical computers are described. Three types of errors are described: Class A errors due to inaccuracies in components. Class B errors due to mathematical incompleteness in the setup and Class C or personal errors. It seems to be desirable to accept certain Class B errors which are known in order to minimize Class A and Class C errors. Designs questions in relation to scaling are discussed. In the second article, cams, resolvers, trigometric function generators and differentials are discussed relative to available scales and error characteristics, in the above classification. The third article deals with multiplication, division, integration differentiation and link mechanisms. The last article describes a number of applications.

F. J. M.

24. Henry Wallman, "An electronic integral transform computer and the practical solution of integral equations," Franklin Inst., *Jn.*, v. 250, 1950, p. 45–61.

A proposed device is described for presenting

$$\int_a^b K(x, t)f(t)dt$$

in the form of a graph of a function of x on a cathode ray tube. $K(x, t)$ is to be obtained by scanning a photographic plate whose opacity corresponds to the value of K at the point x, t. The multiplication of K and f and the value of $f(t)$ itself are to be obtained by using the components due to Macnee.[1] The author shows how the persistence of the image on the screen of the cathode ray tube can be utilized to construct an iterate of the transform and describes also how non-linear transforms in the form

$$\int_a^b K(x, t)k(t, f(t))dt$$

can be obtained.

The paper also lists a variety of applications of such a device. The case in which $K(x, t) = \cos xt$ yields the impulse response of an electrical network. Such a device would readily yield the coefficients of the orthogonal expansion of a function, the Hilbert transform and the convolution integral.

Another set of applications is concerned with the solution of the integral equations. Special cases treated include simultaneous linear algebraic equations, the Volterra equation, the use of Liouville-Neumann series, Fredholm's integral equation of the second kind, the Dirichlet problem for a plane potential and a non-linear problem for a pendulum.

F. J. M.

[1] A. B. Macnee, "A high speed electronic differential analyzer," I.R.E. *Proc.*, v. 37, 1948, p. 1315–1324 [*MTAC*, v. 4, p. 119–120].

NOTES

124. Leslie John Comrie (1893–1950).—This great table maker and pioneer in the art of mechanical computation was born in New Zealand in 1893. He received his early training and a M.A. degree at the University of New Zealand. He saw active service during the first world war with the New Zealand Expeditionary Forces and after the armistice went to University College, London and Cambridge University, where he received his Ph.D. in Astronomy in 1923. After 3 years teaching in the United States at Swarthmore College and Northwestern University he returned to England and the Royal Greenwich Observatory as Deputy Superintendent of H.M. Nautical Almanac Office. He became Superintendent in 1930 and held that post until 1936. Here he introduced modern computing methods which did much to increase the efficiency and productivity of the office. It is this work which brought out his genius for the organization and keen analysis of computing and table preparation for which he later became so famous. He also served brilliantly as secretary of the BAASMTC during 1929–36 and was much concerned with the production of the committee's first six volumes. In 1937 he left the Observatory to devote his entire energy to the development of the Scientific Computing Service, the first enterprise of its kind. The history of this organization is one of lasting achievement and pioneering

effort. Its contributions to the war effort in the 1940's was the source of much justifiable pride to Comrie. He served as cooperative editor of MTAC from 1944–49, and in 1950 was elected Fellow of the Royal Society. His death, Dec. 11, 1950, came as a relief from a lingering affliction.

The reader is referred to *Mathematical Table Makers*[1] for a complete account of Comrie's writings and tables. He is perhaps known most widely as the editor of the third (1930) and later editions of *Barlow's Tables*. His most recent work was the preparation of the monumental two volume edition of *Chambers's Tables*. He delighted in the minute analysis of a computing problem with respect to a given machine and took great pleasure in finding scientific uses for a particular feature of a machine which had been intended by the manufacturer for some trivial commercial application. This exploitation of the commercially available equipment with its built-in, mass-production precision and service was always uppermost in his mind. He had no great enthusiasm for the unreliable specially made computing device. I can recall many friendly arguments with him beginning in 1932 on these two opposing points of view, my side of the argument being, in those days, rather difficult to maintain. His knowledge of numerical analysis was profound and at the same time severely practical. He will be remembered for his "throw back" method of modified differences. On his last visit to America in 1946 he was impressed by what he saw of new computing techniques, but also somewhat dismayed to see to what uses they were put. At one point of the inspection he took me aside to remark: "These people have a terrific amount to learn about computing." This remark is as applicable today as it was when he must have made it (to himself) as a young man at Greenwich. It seems to have been the keynote of this crusading calculator.

D. H. L.

[1] R. C. ARCHIBALD, *Mathematical Table Makers* (The Scripta Mathematica Series no. 3), New York, 1948 [*MTAC*, v. 3, p. 143]. In this work will be found two portraits of Comrie.

125. THE d^2 TEST OF RANDOM DIGITS.—The testing of digits for local randomness, when the digits are to be used in a Monte Carlo method, seems to require a different type of test from the four proposed by KENDALL and BABINGTON-SMITH.[1] Kendall's tests (frequency, serial, gap, poker) apply to random digits as used normally in random sampling, etc.

In a Monte Carlo method, random digits are used to select a random point in the unit square, the digits thus representing the coordinates of a point between (0, 0) and (1, 1).

For two such points, the probability that the square of the distance between them be less than α^2 is given by:[2]

$$P = \pi\alpha^2 - \frac{8\alpha^3}{3} + \frac{\alpha^4}{2},$$

$$\text{for} \quad \alpha^2 = 0.0, 0.1, 0.2, \cdots, 0.9$$

$$P = \tfrac{1}{3} + (\pi - 2)\alpha^2 + 4(\alpha^2 - 1)^{\frac{1}{2}} + \frac{8}{3}(\alpha^2 - 1)^{\frac{3}{2}} - \frac{\alpha^4}{2} - 4\alpha^2 \sec^{-1}\alpha$$

$$\text{for} \quad \alpha^2 = 1.0, 1.1, 1.2, \cdots, 2.0.$$

P is, of course, a continuous function. Discrete values of α^2 are selected for convenience in computation.

These probabilities are, in order:

α^2	P	α^2	P
0.1	.234832	1.1	.985703
0.2	.409805	1.2	.992048
0.3	.549300	1.3	.995788
0.4	.662018	1.4	.997926
0.5	.752987	1.5	.999080
0.6	.825601	1.6	.999652
0.7	.882349	1.7	.999898
0.8	.925163	1.8	.999982
0.9	.955593	1.9	.999999
1.0	.974926	2.0	1.000000

In applying the test, sets of 3 digits are sufficient to represent one coordinate; i.e., 12 digits are selected to give both coordinates of two points.

In a limited test, no significant difference was found when using two-digit coordinates; in other words, the third digit has only a minor effect on the square of the distance. The square of the distance is computed and punched to one decimal place. The distribution of the results is compared to the theoretical by the Chi-squared test. The calculation can be readily carried out on any of the IBM calculators (602, 602A, 604). The distribution of the results can be made in one run on a tabulator equipped with two digit selectors.

This test was carried out on the random digit table of the University of Wisconsin Computing Service (10,000 cards, each bearing 40 digits), making six calculations of d^2 on each card, selecting the columns to be read at random. Results were tabulated for each 1000 cards (6000 d^2's); the χ^2 analysis giving:

Thousand	χ^2	p
1	17.085	.26
2	11.847	.62
3	18.585	.19
4	7.529	.91
5	7.762	.91
6	11.340	.66
7	14.162	.44
8	19.088	.17
9	8.370	.86
10	9.747	.81

using 14 degrees of freedom, since the last 6 classes were lumped together.

Wiring diagrams for the d^2 calculation on the 602A calculating punch and the distribution of results on the 416 or 405 tabulator are available on request.

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A. M. MARK

[1] M. G. KENDALL & B. BABINGTON-SMITH, R. Stat. Soc., *Jn.*, v. 101, 1938, p. 157, and *Supplement*, v. 6, 1939, p. 51.

[2] BENJAMIN WILLIAMSON, *Integral Calculus*. 6th ed., London, 1891, p. 390.

126. On the Calculation of the Square Root by Automatic Computing Machines.—The calculation of the square root of a number on an automatic computing machine not equipped with an internally controlled square root order requires the application of some standard sub-program. It is generally convenient to employ a well-known iterative procedure based upon the formula

$$(1) \qquad R^{(n+1)} = \tfrac{1}{2}[R^{(n)} + N/R^{(n)}],$$

where $R^{(n)}$ is the nth approximation to the square root $R = N^{\frac{1}{2}}$. The efficient application of this procedure requires a systematic method of selecting a zeroth approximation $R^{(0)}$ to the square root R, and the number of iterations required to calculate R to a specified number of significant figures depends largely upon the accuracy of the zeroth approximation.

Let N be a decimal number in the range $1 \leqslant N < 100$. Define ξ by

$$R = \xi + N/10,$$

and let the zeroth approximation $R^{(0)}$ be given by

$$R^{(0)} = \bar{\xi} + N/10,$$

where $\bar{\xi}$ is an average value of ξ. If the number N is distributed uniformly on a logarithmic scale,[1] we may take

$$\bar{\xi} = (\log 100)^{-1} \int_1^{100} (N^{\frac{1}{2}} - N/10) d \log N = 1.759 \sim 2,$$

the one-digit approximation $\bar{\xi} = 2$ being sufficiently accurate. Therefore, we let

$$(2) \qquad R^{(0)} = 2 + N/10.$$

With (1), the first approximation $R^{(1)}$ may be written

$$R^{(1)} = \frac{1}{2}\left[\frac{N}{10} + 2 + \frac{10N}{N + 20}\right].$$

This approximation differs from the root R by an amount $\Delta = R^{(1)} - R$, where

$$\Delta = 1 + \frac{R^2}{20} + \frac{5R^2}{R^2 + 20} - R.$$

In the range $1 \leqslant R \leqslant 10$, Δ is positive. It is equal to 0.25 at $R = 1$, decreases to a minimum of 0.038 at $R = 1.957 \cdots$ and is equal to $0.166 \cdots$ at $R = 10$. Therefore, the choice, (2), of a zeroth approximation leads to a first approximation with at least one correct significant figure. The calculation of the square root by the application of (1) correct to eight significant figures requires a maximum of five iterations and to sixteen significant figures a maximum of six iterations.

If N^* is any positive number such that $N^* = 10^{2k}N$, $1 \leqslant N < 100$, the zeroth approximation $R^{(0)*}$ to the root $R^* = (N^*)^{\frac{1}{2}}$ is

$$R^{(0)*} = 10^k(2 + N/10).$$

This method has been applied in this laboratory to the formulation of a square-root order on control panels for 10-digit arithmetic utilizing the IBM Card-Controlled Electronic Calculator.

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[1] It is reasonable to assume that dimensional numbers will be uniformly distributed on a logarithmic scale if the choice of dimensional units is random. However, this point is unimportant, since this result is comparatively insensitive to the nature of the averaging process.

QUERIES

37. THE SQUARE ROOT METHOD FOR LINEAR EQUATIONS.—In a letter dated 7 Feb. 1950 Mr. H. F. RAINSFORD of Colonial Surveys, Bushy Park, Teddington, England, commenting on the article entitled "The square root method for solving simultaneous linear equations" by J. LADERMAN in *MTAC*, v. 3, p. 13–16, points out that this method was not "probably first discovered by Banachiewicz in 1938" but goes back at least to CHOLESKY whose treatment of the problem was described by BENOÎT[1] in 1924. Can any reader supply an earlier reference to this method?

[1] BENOÎT, "Note sur une méthode de résolution des équations normales provenant de l'application de la méthode des moindres carrés à un système d'équations lineaires en nombre inférieur à celui des inconnues. Application de la méthode a la résolution d'un système define d'equations linéaires," International Geodetic and Geophysical Union, Association of Geodesy, *Bullétin Géodésique*, no. 2, 1924, p. 67–77. An English translation of this article has been kindly supplied by Mr. Rainsford and is available in the UMT FILE.

QUERIES—REPLIES

47. RUSSIAN BESSEL FUNCTION TABLES (Q **25**, v. 3, p. 66).—The volume referred to in this Query, namely: *Tablifsy Znachenii Funktsii Besselia ot Mnimogo Argumenta*, was not published until 1950. [It will be reviewed in the next issue of *MTAC*.]

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CORRIGENDA

V. 4, p. 130 equation (2) *for* superscripts $p + 1$ and $p + 2$ *read* $p + 2$ and $p + 4$.
V. 4, p. 151, l. -4, *for* 23 *read* 33.
V. 4, p. 205 l. 2, *for* by. *read* by:.
V. 4, p. 244, l. 1 *for* on *read* an.
V. 4, p. 255–259. The following omissions and corrections in the Name Index were kindly supplied by S. A. JOFFE:

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